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(英文版)

数学物理方程

——傅里叶分析及其应用

Fourier Analysis and Its Applications

(美) Gerald B. Folland 著

机械工业出版社
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序

本书的特点是把傅里叶分析的方法作为解决物理、工程问题的工具来进行讲述的。作者力图使这本书既适合工程技术人员也对纯数学工作者有益。

作者从介绍最基本的数学物理方程问题的背景开始，然后讲述傅里叶级数的最基本理论。在处理收敛问题时只涉及逐段光滑函数，然后引入 L^2 理论，此时以承认勒贝格积分论为前提。接下来讲述热传导方程、波动方程等微分方程的边值问题。并介绍非常有用的特殊函数——贝塞尔函数和基本的正交多项式：勒让得多项式、埃尔米特多项式、拉盖尔多项式等。接着讲述傅里叶变换、拉普拉斯变换。然后介绍近代发展起来的十分有用的广义函数理论和广义函数的傅里叶变换。最后一章在前一章的基础上阐述微分算子的格林函数的知识。

本书以傅里叶分析为主线，以解决典型的数学物理方程为目标来展开相关的数学理论，是一本数学物理方程课程的教材。它适应我国综合大学和工科大学本科高年级及研究生数学、物理及工程专业的学生学习。学习此书的要求是具备微积分、常微分方程和最基本的复变函数基础。

从书中的内容来看，作者的数学造诣是比较深的，因为在结合实际方程讲述数学理论时，是明白、清楚的。数学物理方程这门课本身就具有很强的综合性、应用性，理论难度并不大，但涉及知识领域广，基础要求宽。国内好的数学物理方程教材并不多。

本书能适应国内教学的需要，而且此书以傅里叶分析为主线的结构和比较清晰严谨的逻辑安排是很值得称道的。当然，从纯数学的角度来看，此书的内容的理论深度不大，所以可读性较强，有广泛的读者。

北京师范大学

王昆扬

PREFACE

This book is intended for students of mathematics, physics, and engineering at the advanced undergraduate level or beyond. It is primarily a text for a course at the advanced undergraduate level, but I hope it will also be useful as a reference for people who have taken such a course and continue to use Fourier analysis in their later work. The reader is presumed to have (i) a solid background in calculus of one and several variables, (ii) knowledge of the elementary theory of linear ordinary differential equations (i.e., how to solve first-order linear equations and second-order ones with constant coefficients), and (iii) an acquaintance with the complex number system and the complex exponential function $e^{x+iy} = e^x(\cos y + i \sin y)$. In addition, the theory of analytic functions (power series, contour integrals, etc.) is used to a slight extent in Chapters 5, 6, 7, and 9 and in a serious way in Sections 8.2, 8.4, 8.6, 10.3, and 10.4. I have written the book so that lack of knowledge of complex analysis is not a serious impediment; at the same time, for the benefit of those who do know the subject, it would be a shame not to use it when it arises naturally. (In particular, the Laplace transform without analytic functions is like Popeye without his spinach.) At any rate, the facts from complex analysis that are used here are summarized in Appendix 2.

The subject of this book is the whole circle of ideas that includes Fourier series, Fourier and Laplace transforms, and eigenfunction expansions for differential operators. I have tried to steer a middle course between the mathematics-for-engineers type of book, in which Fourier methods are treated merely as a tool for solving applied problems, and the advanced theoretical treatments aimed at pure mathematicians. Since I thereby hope to please both the pure and the applied factions but run the risk of pleasing neither, I should give some explanation of what I am trying to do and why I am trying to do it.

First, this book deals almost exclusively with those aspects of Fourier analysis that are useful in physics and engineering rather than those of interest only in pure mathematics. On the other hand, it is a book on *applicable* mathematics rather than *applied* mathematics: the principal role of the physical applications herein is to illustrate and illuminate the mathematics, not the other way around. I have refrained from including many applications whose principal conceptual content comes from Subject X rather than Fourier analysis, or whose appreciation requires specialized knowledge from Subject X; such things belong more properly in a book on Subject X where the background can be more fully explained. (Many of my favorite applications come from quantum physics, but in accordance with this principle I have mentioned them only briefly.) Similarly, I have not worried too much about the physical details of the applications studied here. For example, when I think about the 1-dimensional heat equation I usually envision a long thin rod, but one who prefers to envision a 3-dimensional slab whose temperature varies only along one axis is free to do so; the mathematics is the same.

Second, there is the question of how much emphasis to lay on the theoretical aspects of the subject as opposed to problem-solving techniques. I firmly believe that theory — meaning the study of the ideas underlying the subject and the reasoning behind the techniques — is of intellectual value to everyone, applied or pure. On the other hand, I do not take "theory" to be synonymous with "logical rigor." I have presented complete proofs of the theorems when it is not too onerous to do so, but I often merely sketch the technical parts of an argument. (If the technicalities cannot easily be filled in by someone who is conversant with such things, I usually give a reference to a complete proof elsewhere.) Of course, where to draw the line is a matter of judgment, and I suppose nobody will be wholly satisfied with my choices. But those instructors who wish to include more details in their lectures are free to do so, and readers who tire of a formal argument have only to skip to the end-of-proof sign ∎. Thus, the book should be fairly flexible with regard to the level of rigor its users wish to adopt.

One feature of the theoretical aspect of this book deserves special mention. The development of Lebesgue integration and functional analysis in the period 1900–1950 has led to enormous advances in our understanding of the concepts underlying Fourier analysis. For example, the completeness of L^2 and the shift from pointwise convergence to norm convergence or weak convergence simplifies much of the discussion of orthonormal bases and the validity of series expansions. These advances have usually not found their way into application-oriented books because a rigorous development of them necessitates the building of too much machinery. However, most of this machinery can be ignored if one is willing to take a few things on faith, as one takes the intermediate value theorem on faith in freshman calculus. Accordingly, in §3.3–4 I assert the existence of an improved theory of integration, the Lebesgue integral, in the context of which one has (i) the completeness of L^2, (ii) the fact that "nice" functions are dense in L^2, and (iii) the dominated convergence theorem. I then proceed to use these facts without further ado. (The dominated convergence theorem, it should be noted, is a wonderful tool even in the context of Riemann integrable functions.) Later, in Chapter 9, I develop the theory of distributions as linear functionals on test functions, the motivation being that the value of a distribution on a test function is a smeared-out version of the value of a function at a point. Discussion of functional-analytic technicalities (which are largely irrelevant at the elementary level) is reduced to a minimum.

With the exception of the prerequisites and the facts about Lebesgue integration mentioned above, this book is more or less logically self-contained. However, certain assertions made early in the book are established only much later:

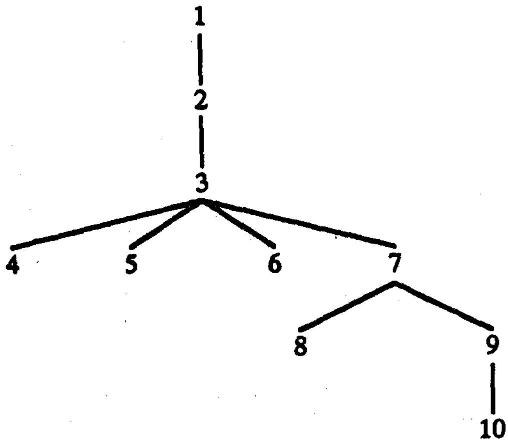
(i) The completeness of the eigenfunctions of regular Sturm-Liouville problems is stated in §3.5 and proved, in the case of separated boundary conditions, in §10.3.

(ii) The asymptotic formulas for Bessel functions given in §5.3 are proved via Watson's lemma in §8.6.

(iii) The proofs of completeness of Legendre, Hermite, and Laguerre polynomials in Chapter 6 rely on the Weierstrass approximation theorem and the Fourier inversion theorem, proved in Chapter 7.

(iv) The discussion of weak solutions of differential equations in §9.5 justifies many of the formal calculations with infinite series in the earlier chapters.

Thus, among the applications of the material in the later part of the book is the completion of the theory developed in the earlier part.



Chapter Dependence Diagram

The main dependences among the chapters are indicated in the accompanying diagram, but a couple of additional comments are in order.

First, there are some minor dependences that are not shown in the diagram. For example, a few paragraphs of text and a few exercises in Sections 6.3, 7.5, 8.1, and 8.6 presuppose a knowledge of Bessel functions, but one can simply omit these bits if one has not covered Chapter 5. Also, the discussion of techniques in §4.1 is relevant to the applied problems in later chapters, particularly in §5.5.

Second, although Chapter 10 depends on Chapter 9, except in §10.2 the only part of distribution theory needed in Chapter 10 is an appreciation of delta functions on the real line and the way they arise in derivatives of functions with jump discontinuities. Hence, one could cover Sections 10.1 and 10.3–4 after an informal discussion of the delta function, without going through Chapter 9.

There is enough material in this book for a full-year course, but one can also select various subsets of it to make shorter courses. For a one-term course one could cover Chapters 1–3 and then select topics *ad libitum* from Chapters 4–7. (If one wishes to present some applications of Bessel functions without discussing the theory in detail, one could skip from the recurrence formulas in §5.2 to the statement of Theorem 5.3 at the end of §5.4 without much loss of continuity.) I have taught a one-quarter (ten-week) course from Chapters 1–5 and a sequel to it from Chapters 7–10, omitting a few items here and there.

One further point that instructors should keep in mind is the following. Most of the book deals with rather concrete ideas and techniques, but there are two places where concepts of a more general and abstract nature are discussed in a serious way: Chapter 3 (L^2 spaces, orthogonal bases, Sturm-Liouville problems) and Chapter 9 (functions as linear functionals, generalized functions). These parts are likely to be difficult for students who have had little experience with abstract mathematics, and instructors should plan their courses accordingly.

Fourier analysis and its allied subjects comprise an enormous amount of mathematics, about which there is much more to be said than is included in this book. I hope that my readers will find this fact exciting rather than dismaying. Accordingly, I have included a sizable although not exhaustive bibliography of books and papers to which the reader can refer for more information on things that are touched on lightly here. Most of these references should be reasonably accessible to the students for whom this book is primarily intended, but a few of them are of a considerably more advanced nature. This is inevitable; the topics in this book impinge on a lot of sophisticated material, and the full story on some of the things discussed here (singular Sturm-Liouville problems, for instance) cannot be told without going to a deeper level. But these advanced references should be of use to those who have the necessary background, and may at least serve as signposts to those who have yet to acquire it.

I am grateful to my colleagues Donald Marshall, Douglas Lind, Richard Bass, and James Morrow and to the students in our classes for pointing out many mistakes in the first draft of this book and suggesting a number of improvements. I also wish to thank the following reviewers for their helpful suggestions in revising the manuscript: Giles Auchmuty, University of Houston; James Herod, Georgia Institute of Technology; Raymond Johnson, University of Maryland; Francis Narcowich, Texas A & M University; Juan Carlos Redondo, University of Michigan; Jeffrey Rauch, University of Michigan; Jesus Rodriguez, North Carolina State University; and Michael Vogelius, Rutgers University.

Gerald B. Folland

CONTENTS

CHAPTER 1
OVERTURE

The subject of this book is *Fourier analysis,* which may be described as a collection of related techniques for resolving general functions into sums or integrals of simple functions or functions with certain special properties. Fourier analysis is a powerful tool for many problems, and especially for solving various differential equations of interest in science and engineering. The purpose of this introductory chapter is to provide some background concerning partial differential equations. Specifically, we introduce some of the basic equations of mathematical physics that will provide examples and motivation throughout the book, and we discuss a technique for solving them that leads directly to problems in Fourier analysis.

At the outset, let us present some notations that will be used repeatedly. The real and complex number systems will be denoted by **R** and **C**, respectively. We shall be working with functions of one or several real variables $x_1, \ldots, x_n$. We shall denote the ordered n-tuple $(x_1, \ldots, x_n)$ by $\mathbf{x}$ and the space of all such ordered n-tuples by $\mathbf{R}^n$.

In most of the applications, n will be 1, 2, 3, or 4, and the variables x_j will denote coordinates in one, two, or three space dimensions, together with time. In this situation we shall usually write x, y, z instead of x_1, x_2, x_3 for the spatial variables, and we shall denote the time variable by t. Moreover, we shall use the common subscript notation for partial derivatives:

$$u_x = \frac{\partial u}{\partial x}, \qquad u_{xx} = \frac{\partial^2 u}{\partial x^2} \qquad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \qquad \text{etc.}$$

A function f of one real variable is said to be of class $C^{(k)}$ on an interval I if its derivatives $f', \ldots, f^{(k)}$ exist and are continuous on I. Similarly, a function of n real variables is said to be of class $C^{(k)}$ on a set $D \subset \mathbf{R}^n$ if all of its partial derivatives of order $\leq k$ exist and are continuous on D. If the function possesses continuous derivatives of all orders, it is said to be of class $C^{(\infty)}$.

Finally, we use the common notation with square and round brackets for closed and open intervals in the real line **R**:

$$[a, b] = \{x : a \leq x \leq b\}, \qquad (a, b) = \{x : a < x < b\},$$
$$[a, b) = \{x : a \leq x < b\}, \qquad (a, b] = \{x : a < x \leq b\}.$$

1.1 Some equations of mathematical physics

In order to understand the significance of the ideas as they arise, it will be useful to have a few physical applications in mind as examples of the sort of problems we are trying to solve. Accordingly, we begin with a brief and informal discussion of some of the basic partial differential equations of classical mathematical physics. These equations all involve a fundamental differential operator known as the Laplacian, which is defined as follows. If u is a function of the real variables $x_1, \ldots, x_n$ of class $C^{(2)}$, the **Laplacian** of u is the function $\nabla^2 u$ defined by

$$\nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}. \tag{1.1}$$

The first of the equations we shall study is the **wave equation**:

$$u_{tt} = \frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u. \tag{1.2}$$

Here u represents a wave traveling through an n-dimensional medium—where, in practice, n will usually be 1, 2, or 3. More precisely, $x_1, \ldots, x_n$ are the coordinates of a point $\mathbf{x}$ in the medium; t is the time; c is the speed of propagation of waves in the medium; and $u(\mathbf{x}, t)$ is the amplitude of the wave at position $\mathbf{x}$ and time t.

The wave equation provides a reasonable mathematical model for a number of physical processes, such as the following:

(a) Vibrations of a stretched string, such as a guitar string.
(b) Vibrations of a column of air, such as an organ pipe or clarinet.
(c) Vibrations of a stretched membrane, such as a drumhead.
(d) Waves in an incompressible fluid, such as water.
(e) Sound waves in air or other elastic media.
(f) Electromagnetic waves, such as light waves and radio waves.

The number n of spatial dimensions is 1 in examples (a) and (b), 2 in examples (c) and (d) (since the waves appear on the *surface* of the water), and 3 in examples (e) and (f). In (a), (c), and (d), u represents the transverse displacement of the string, membrane, or fluid surface; in (b) and (e), u represents the longitudinal displacement of the air; and in (f), u is any of the components of the electromagnetic field.

We shall not attempt to derive the wave equation from physical principles here, since each of the preceding examples involves different physics. Examples (a) and (f) are explained in Appendix 1; discussions of the others may be found, for example, in Ingard [32]* and Taylor [51]. We should point out, however, that in most cases the derivation involves making some simplifying assumptions. Hence, the wave equation gives only an approximate description of the actual physical process, and the validity of the approximation will depend on whether

* Numbers in brackets refer to the bibliography at the end of the book.

certain physical conditions are satisfied. For instance, in example (a) the vibrations should be small enough so that the string is not stretched beyond its limits of elasticity. In example (f) it follows from Maxwell's equations, the fundamental equations of electromagnetism, that the wave equation is satisfied *exactly* in regions containing no electric charges or currents — but of course the assumption of no charges or currents can only be approximately valid in the real world. (Of course, it is precisely the fact that the wave equation is only an approximation that allows it to be a useful model in so many different situations!)

The next basic differential equation on our list is the **heat equation**:

$$u_t = k\nabla^2 u. \tag{1.3}$$

This equation describes the diffusion of thermal energy in a homogeneous material (that is, one whose composition does not change from point to point). As in the wave equation, the variables x_j are spatial coordinates and t is time, but now $u(\mathbf{x}, t)$ is the *temperature* at a position $\mathbf{x}$ and time t, and k is a constant called the "thermal diffusivity" of the material. A brief derivation is given in Appendix 1. As for the number of spatial variables, the case $n = 3$ is the most fundamental from the physical point of view, but the cases $n = 1$ and $n = 2$ are also of interest as models of situations where the heat flow is practically all in one or two directions. For example, the heat equation with $n = 1$ can be used to describe heat flow along a wire or rod, provided that heat flow in directions perpendicular to the axis of the rod can be neglected. It can also be used to describe heat flow in a slab of material, such as a wall separating two rooms, where only the heat flow from one room toward the other (as opposed to flow in directions parallel to the wall) is significant.

Two warnings: (i) The heat equation can be used to model heat flow in both solids and fluids (liquids and gases), but in the latter case it does *not* take any account of the phenomenon of convection; that is, it will provide a reasonable model only if conditions are such as to exclude any macroscopic currents in the fluid. (ii) The heat equation is not a fundamental law of physics, and it does not give reliable answers at very low or very high temperatures. In particular, it is obvious that if u is a solution then so is $u + c$ for any constant c; thus the heat equation does not recognize the existence of absolute zero!

The heat equation can also be used to model other diffusion processes. For example, if a drop of red dye is placed in a body of water, the dye will gradually spread out and permeate the entire body. If convection effects are negligible, equation (1.3) will describe the diffusion of the dye through the water ($u(\mathbf{x}, t)$ now being the concentration of dye at position $\mathbf{x}$ and time t).

Next, we come to the **Laplace equation**:

$$\nabla^2 u = 0. \tag{1.4}$$

Laplace's equation arises in a number of different contexts. It is satisfied by the electrostatic potential in any region containing no electric charge, and by the gravitational potential in any region containing no mass. It is also the equation that governs standing waves and steady-state heat distributions — that is, solutions of the wave equation and the heat equation that are independent of time. We shall meet other applications of it later on.

Partial differential equations such as the ones discussed above typically have solutions in such great abundance that there is no reasonable way of giving an explicit description of all of them. The most common way of pinning down a particular solution is to impose some boundary conditions. Different types of differential equations require different types of boundary conditions, and the particular conditions that are appropriate for a given physical problem will depend on the particular physical situation. The physics is generally a good guide to the mathematics: "reasonable" physical conditions usually lead to "reasonable" mathematical problems.

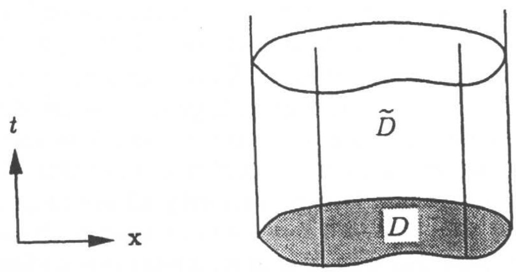


FIGURE 1.1. The region D in $\mathbf{x}$-space and the region $\tilde{D}$ in $\mathbf{x}t$-space.

These matters may best be explained by examining a few examples. Let us consider the heat equation: suppose we are interested in studying the diffusion of heat in a body that occupies a bounded region D of $\mathbf{x}$-space, given the initial temperature distribution in the body. That is, we wish to solve the heat equation (1.3) in the region

$$\tilde{D} = \left\{(\mathbf{x}, t) : \mathbf{x} \in D, \quad t > 0\right\}$$

of $(\mathbf{x}, t)$-space subject to the initial condition

$$u(\mathbf{x}, 0) = f(\mathbf{x}), \tag{1.5}$$

where $f(\mathbf{x})$ is the temperature distribution at time $t = 0$. (See Figure 1.1.) Equation (1.5) is a condition on u on the "horizontal" part of the boundary of $\tilde{D}$, but it is not enough to specify u completely; we also need a boundary condition on the "vertical" part of the boundary to tell what happens to the heat when it reaches the boundary surface S of the spatial region D. Here the particular physical conditions at hand must be our guide. One reasonable assumption is that S is held at a constant temperature u_0 (for example, by immersing the body in a bath of ice water), thus:

$$u(\mathbf{x}, t) = u_0 \quad \text{for } \mathbf{x} \in S, \ t > 0. \tag{1.6}$$

Another reasonable assumption is that D is insulated, so that no heat can flow in or out across S. Mathematically, this amounts to requiring the *normal derivative* of u along the boundary S to vanish:

$$(\nabla u \cdot \mathbf{n})(\mathbf{x}, t) = 0 \quad \text{for } \mathbf{x} \in S,\ t > 0. \tag{1.7}$$

Here ∇u is the gradient of u in $\mathbf{x}$ and $\mathbf{n}$ is the unit outward normal vector to S (and we are implicitly assuming that the surface S is smooth, so that $\mathbf{n}$ is well-defined). A more realistic assumption than either (1.6) or (1.7) is that the region outside D is held at a constant temperature u_0, and the rate of heat flow across the boundary S is proportional to the difference in temperatures on the two sides:

$$(\nabla u \cdot \mathbf{n})(\mathbf{x}, t) + a\Big(u(\mathbf{x}, t) - u_0\Big) = 0 \quad \text{for } \mathbf{x} \in S,\ t > 0. \tag{1.8}$$

This is **Newton's law of cooling,** and $a > 0$ is the proportionality constant. The conditions (1.6) and (1.7) may be regarded as the limiting cases of (1.8) as $a \to \infty$ or $a \to 0$.

At any rate, it turns out that the initial condition (1.5) together with any one of the boundary conditions (1.6), (1.7), or (1.8) leads to a *well-posed* problem: one having a unique solution that depends continuously (in some appropriate sense) on the initial data f. The same discussion is also valid for the heat equation in one or two space dimensions. (In one space dimension, the "region" D is just an interval in the x-axis, and the "normal derivative" $\nabla u \cdot \mathbf{n}$ is just u_x at the right endpoint and $-u_x$ at the left endpoint.)

A similar analysis applies to boundary value problems for the wave equation (1.2), with one significant difference: the wave equation is second-order in the time variable t, whereas the heat equation is only first-order in t. For this reason, in solving the wave equation it is appropriate to specify not only the initial values of u as in (1.5) but also the initial velocity u_t:

$$u(\mathbf{x}, 0) = f(\mathbf{x}), \qquad u_t(\mathbf{x}, 0) = g(\mathbf{x}) \quad \text{for } \mathbf{x} \in D. \tag{1.9}$$

The imposition of the initial conditions (1.9) together with a boundary condition of the form (1.6), (1.7), or (1.8) leads to a unique solution of the wave equation. For example, to analyze the motion of a vibrating string of length l that is fixed at both endpoints, we take the "region" D to be the interval $[0, l]$ on the x-axis and solve the one-dimensional wave equation with boundary conditions (1.6) (where $u_0 = 0$) and (1.9):

$$\begin{aligned} u_{tt} = c^2 u_{xx}, \qquad & u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x) \quad \text{for } 0 < x < l, \\ & u(0,t) = u(l,t) = 0 \quad \text{for } t > 0. \end{aligned}$$

Remark: The "velocity" u_t is not the same as the constant c in the wave equation. c is the speed of propagation of the wave along the string, whereas u_t is the rate of change of the displacement of a particular point on the string. (The same is true for waves in media other than strings.)