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Kac-Moody Algebras and Their Representations

Xiaoping Xu

(卡茨－穆迪代数及其表示)

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Dedicated to My Parents

Preface

In order to study infinite-dimensional Lie algebras with root space decomposition as finite-dimensional simple Lie algebras, Victor Kac and Robert Moody independently introduced Lie algebras associated with generalized Cartan matrices, so-called "Kac-Moody algebras" in later 1960s. In last near forty years, these algebras have played important roles in the other mathematical fields such as combinatorics, number theory, topology, integrable systems, operator theory, quantum stochastic process, and in quantum field theory of physics.

There have been several books on Kac-Moody algebras. The most authoritative and influential one may be the monograph "Infinite-dimensional Lie algebras" by Victor Kac. Our purpose of writing this book is to help the readers to better understand Kac's book. This book was written based on my lecture notes in Kac-Moody algebras taught in Chinese Academy of Mathematics and System Sciences in 2005, 2006. We have tried to give the details that Kac's book lacks and correct some mistakes. In many occasions, we have reorganized the materials. For instance, we have added detailed vertex operator and free fermionic field representations of affine Kac-Moody algebras. Of course, we have also deleted some materials in Kac's book which do not seem so important to students. Nevertheless, our book contains most of fundamental results in Kac-Moody algebras. Needless to say, this book is not a replacement of Kac's book, but a new choice of textbooks in the field to researchers and students.

I would like to thank my friends Prof. Shaobin Tan and Prof. Yucai Su for their encouragement of writing this book. I am also very grateful to my students Li Luo and Yufeng Zhao, and to Yan Wang (a graduate student at Nan Kai University) for their careful proof reading of the initial manuscript and pointing out numerous typos and errors.

Xiaoping Xu
2006, Beijing

Notational Conventions

$\mathbb{C}$	the field of complex numbers.
$\overline{i, i+j}$	$\{i, i+1, i+2, \cdots, i+j\}$, an index set.

$\delta_{i,j} = 1$ if $i = j$, 0 if $i \neq j$.

$\mathbb{Z}$	the ring of integers.
$\mathbb{Z}_+$	$\{0, 1, 2, 3, \cdots\}$, the set of natural numbers
$\mathbb{Q}$	the field of rational numbers.
$\mathbb{R}$	the field of real numbers.
Π	the set of positive simple roots.
Δ	the set of roots.
Ω	generalized Casimier operator
$W(A)$	the Weyl group.
r_i	the ith simple reflection.

Contents

Introduction

Mathematics is a logical science. Lie algebra is not a mysterious subject. It can be viewed as "advanced linear algebra" in a certain sense. In linear algebra, we mainly study vector spaces and single linear transformation. Recall that an $n \times n$ Jordan block is a matrix of the form:

$$J_{n,\lambda} = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ \vdots & \ddots & \lambda & \ddots & 0 \\ 0 & \cdots & 0 & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}, \tag{0.1}$$

where $\lambda \in \mathbb{C}$, the field of complex numbers. A fundamental theorem in linear algebra says that any linear transformation T on a finite-dimensional vector space over $\mathbb{C}$ takes the *Jordan form* :

$$T = \begin{pmatrix} J_{n_1,\lambda_1} & 0 & \cdots & 0 \\ 0 & J_{n_2,\lambda_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_{n_k,\lambda_k} \end{pmatrix} \tag{0.2}$$

with respect to a certain basis.

Lie algebra is a field of studying a vector space V and a subspace $\mathcal{G}$ of linear transformations on V such that

$$AB - BA \in \mathcal{G} \qquad \text{if } A, B \in \mathcal{G}, \tag{0.3}$$

where $\mathcal{G}$ is called a *Lie algebra* and the linear transformation $AB - BA$ is called the *commutator* of linear transformations, denoted as $[A, B]$. "Simple Lie algebra" and "irreducible modules" in Lie theory are generalizations of the Jordan blocks. "Completely reducibility" is exactly a generalization of the Jordan form.

Lie algebra is not purely an abstract mathematics but a fundamental tool of studying symmetries in the world. In fact, Norwegian mathematician Sorphus Lie introduced Lie algebra in later 19th century in order to study the symmetry of differential equations. For instance, we have the following theorem on a system of ordinary linear differential equations:

Theorem of Lie and Scheffers *The general solution $\vec{x}(t)$ of the system of equations:*

$$\frac{dx_i(t)}{dt} = f_i(\vec{x}, t), \qquad i = 1, 2, \cdots, n, \tag{0.4}$$

can be expressed as a function of m particular solutions and n significant constants

$$\vec{x}(t) = \vec{s}(\vec{x}_1(t), \cdots, \vec{x}_m(t), c_1, \cdots, c_n) \tag{0.5}$$

if and if only

$$f_i(\vec{x}, t) = \sum_{j=1}^{m} \xi_{i,j}(\vec{x}) g_j(t), \tag{0.6}$$

and the differential operators

$$\left\{ \sum_{i=1}^{n} \xi_{i,j}(\vec{x}) \partial_{x_i} \mid j = 1, 2, \cdots, m \right\} \tag{0.7}$$

span a Lie algebra of dimension m with respect to the commutator.

In general, a differential equation can be solved explicitly just because it has a certain symmetry related to Lie algebras.

Lie algebras are the infinitesimal structures (bones) of Lie groups, which are symmetric manifolds. Stochastic Leowner evolution is connected to Lie algebras with one-variable structure via conformal field theory (cf. [HP], [LSW], [So]). The controllability property of the unitary propagator of an N-level quantum mechanical system subject to a single control field can be described in terms of the structure theory of semisimple Lie algebras (cf. [DPRR]). Moreover, Lie algebras were used to explain the degeneracies encountered in the genetic code as the result of a sequence of symmetry breakings that have occurred during its evolution (cf. [HH]).

The initial of quantum physics is the *uncertainty principle*, which says that one can not measure the momentum and position of a particle at the same time. If we denote by ΔP the error of the momentum and by Δx the error of position, then

$$\Delta P \cdot \Delta x > \hbar, \tag{0.8}$$

where $\hbar$ is called the *Plunk constant.* Let $\mathbb{C}[x]$ be the algebra of polynomials in x. Define the left multiplication operator $L_x : \mathbb{C}[x] \to \mathbb{C}[x]$ by

$$L_x(f(x)) = xf(x) \qquad \text{for } f(x) \in \mathbb{C}[x]. \tag{0.9}$$

Represent the momentum by the operator d/dx and the position by L_x. The uncertainty can be mathematically interpreted as that the two operators d/dx and L_x can not have a common eigenvector, due to the non-commutativity:

$$\frac{d}{dx} \circ L_x - L_x \circ \frac{d}{dx} = \mathrm{Id}_{\mathbb{C}[x]}. \tag{0.10}$$

In quantum physics, a physical entity becomes an operator on a certain Hilbert space of physical states, which are probability functions. The Hilbert space are usually symmetric with respect to a certain Lie algebra. A *quantum field* is an operator-valued function on the Hilbert space. It turns out that the coefficients of certain quantum fields are connected to "affine Kac-Moody algebras". Throughout this book, all the vector spaces and algebras are assumed over $\mathbb{C}$.

For a vector space V, we denote by V^* the space of linear functions on V. The motivation of introducing Kac-Moody algebra was essentially to study the following type of Lie algebra $\mathcal{G}$: (1) $\mathcal{G}$ contains a subspace H such that

$$\mathcal{G} = \bigoplus_{\alpha \in H^*} \mathcal{G}_\alpha, \qquad \mathcal{G}_\alpha = \{u \in \mathcal{G} \mid [h,u] = \alpha(h)u \text{ for } h \in H\}, \tag{0.11}$$

and $\mathcal{G}_0 = H$; (2) $\bigoplus_{0 \neq \alpha \in H^*} \mathcal{G}_\alpha$ is contained in the subalgebra $\mathcal{G}'$ generated by $\{\mathcal{G}_{\pm\alpha_1}, \mathcal{G}_{\pm\alpha_2}, \cdots, \mathcal{G}_{\pm\alpha_n}\}$ with $\dim \mathcal{G}_{\pm\alpha_i} = 1$; (3) the subalgebra $\mathcal{G}_+$ generated by $\{\mathcal{G}_{\alpha_1}, \mathcal{G}_{\alpha_2}, \cdots, \mathcal{G}_{\alpha_n}\}$ dose not contain a nonzero proper subspace U such that

$$[u,v] \subset U \qquad \text{for } \ u \in U, v \in \mathcal{G} \tag{0.12}$$

and neither does the subalgebra $\mathcal{G}_-$ generated by $\{\mathcal{G}_{-\alpha_1}, \mathcal{G}_{-\alpha_2}, \cdots, \mathcal{G}_{-\alpha_n}\}$. Such an idea was also used by Li and the author [LX] to characterize "lattice vertex operator algebras".

A Lie algebra $\mathcal{G}$ is called *simple* if it does not contain a nonzero proper subspace U such that

$$[u,v] \in U \qquad \text{for } \ u \in U, \ v \in \mathcal{G}. \tag{0.13}$$

The works of Killing and Cartan showed that any finite-dimensional simple Lie algebra is of the above type. For a Lie algebra $\mathcal{G}$ and $u \in \mathcal{G}$, the *adjoint operator* ad u is defined by

$$(\mathrm{ad}\, u)(v) = [u,v] \qquad \text{for } \ v \in \mathcal{G}. \tag{0.14}$$

Chevalley proved that a finite-dimensional simple Lie algebra $\mathcal{G}$ has generators $e_i \in \mathcal{G}_{\alpha_i}$, $f_i \in \mathcal{G}_{-\alpha_i}$ and $h_i \in H$ for $i = 1, 2, \cdots, n$ such that

$$[h_i, h_j] = 0, \ \ [h_i, e_j] = a_{i,j} e_j, \ \ [h_i, f_j] = -a_{i,j} f_j, \ \ [e_i, f_j] = \delta_{i,j} h_j, \tag{0.15}$$

$$(\text{ad}\, e_i)^{1-a_{i,r}}(e_r) = 0, \quad (\text{ad}\, f_i)^{1-a_{i,r}}(f_r) = 0, \qquad i \neq r, \tag{0.16}$$

where the matrix $A = (a_{i,j})_{n\times n}$ is the *Cartan matrix* whose entries are integers satisfying:

$$a_{i,i} = 2, \quad a_{i,j} \leqslant 0, \quad a_{i,j} = 0 \Leftrightarrow a_{j,i} = 0, \quad i \neq j. \tag{0.17}$$

Serre [Sj] showed that (0.15) and (0.16) are indeed the defining relations of $\mathcal{G}$.

An $n \times n$ integer matrix A satisfying (0.17) was called a *generalized Cartan matrix* by Kac [Kv1] and Moody [Mr1]. Moreover, they used (0.15) and (0.16) to construct a Lie algebra $\mathcal{G}(A)$ of the type indicated earlier, so-called *Kac-Moody algebra.* The matrix A is called *symmetrizable* if there exists an invertible diagonal matrix D such that DA is symmetric. The whole business in Kac-Moody algebras took off under the assumption that A is symmtrizable. Under this assumption, the Lie algebra $\mathcal{G}(A)$ has a nondegenerate invariant bilinear form, by which a generalized Casimier operator is obtained. The structure and highest-weight representation theories were established by using the generalized Casimier operator and Weyl group. In particular, an analogue of the Weyl character formula was obtained by Kac [Kv2] for integrable highest-weight irreducible modules.

The fundamental difference between Kac-Moody algebras of infinite type and finite-dimensional simple Lie algebras is that there are roots which are not conjugated to simple roots, that is, *imaginary roots.* It turns out that imaginary roots are exactly the elements in the root lattice with non-positive square norm. The restriction of a highest-weight modules to an imaginary root subalgebra is isomorphic to a direct sum of its Verma modules. When A is indecomposable and of co-rank one, the algebra $\mathcal{G}(A)$ is called an *affine Kac-Moody algebra.* It turns out that affine Kac-Moody algebras have natural loop-algebra realizations, which imply by Sugawara operators that they are conformal invariant. In fact, they appeared in physics as "current algebras". The denominator identities of affine Kac-Moody algebras are exactly the well-known Macdonald's identities (cf. [Mi]). The characters of integrable highest-weight modules of affine Kac-Moody algebras satisfy certain modular transformation properties. In particular, the q-dimensions of the subspaces of imaginary root strings in the modules are modular forms. Affine Kac-Moody algebras are also symmetries of some integrable systems.

Lepowsky and Wilson [LW1] introduced vertex operators in order to study the explicit structure of integrable highest-weight modules of affine Kac-Moody algebras. They [LW2, LW3] used these operators to prove the famous Rogers-Ramanujan identities. Frankel [Fi], and Feingold and Frenkel

[FF2] gave free fermionic field realizations of some integrable highest-weight modules of affine Kac-Moody algebras. We refer [Kv3] for the more detailed history in Kac-Moody algebras and credit countings.

Chapter 1 is mainly the structure theory of Kac-Moody algebras. First we use (0.15) to define the Lie algebra $\mathcal{G}(A)$ associated with any square matrix A without zero rows and columns. When A is a generalized Cartan matrix, the algebra $\mathcal{G}(A)$ was originally introduced by Kac [Kv1] and Moddy [Mr1] independently. This more general settings is used later to prove the defining relations of Kac-Moody algebras. When A is symmetrizable, we construct a nondegenerate invariant bilinear form of the Lie algebra $\mathcal{G}(A)$. By this form, we obtain a generalized Casimir operator, which plays a fundamental role later in the highest-weight representation theory of Kac-Moody algebras. Furthermore, the Weyl groups of Kac-Moody algebras are introduced. A certain classification of generalized Cartan matrices is given. Real and imaginary roots are defined and characterized.

Among Kac-Moody algebras, the simplest infinite-dimensional algebras are those of affine type, that is, the associated generalized Cartan matrices are of co-rank one. In Chapter 2, we give a more detailed study on affine Kac-Moody algebras. First we completely determine the roots and Weyl groups of affine Kac-Moody algebras. Then we give loop algebra realizations of untwisted affine Kac-Moody algebras. Furthermore, we realize twisted affine Kac-Moody algebras as the subalgebras of untwisted affine Kac-Moody algebras fixed by a certain automorphism.

Chapter 3 is to develop a general representation theory of Kac-Moody algebras. First, we give a highest-weight representation theory for the Lie algebra $\mathcal{G}(A)$. The defining relations of the Kac-Moody algebra associated with a symmetrizable generalized Cartan matrix is then proved by using the highest-weight representation theory. Moreover, we derive the character formulas of integrable highest-weight irreducible modules of the Kac-Moody algebras associated with symmetrizable generalized Cartan matrices. The denominator identity and q-dimension formula are also given. Furthermore, we investigate the weights of integrable highest-weight irreducible modules of Kac-Moody algebras. We find the necessary and sufficient condition for highest-weight irreducible modules of Kac-Moody algebras to be unitary. The actions of imaginary root vectors on highest-weight irreducible modules of Kac-Moody algebras are determined. We deduce some combinatorial formulas from the denominator identity.

The root multiplicities and Weyl groups of Kac-Moody algebras are not known in general except those of finite or affine type. In Chapter 4, first we show that the denominator identities imply Macdonald identities. Then we characterize the weights of integrable highest-weight irreducible modules of affine Kac-Moody algebras and prove some of their properties. The formal characters are determined in terms of theta functions and q-dimensions of the subspaces of imaginary root strings. Moreover, we show that the Virasoro algebra naturally appears in derivations of the loop algebra realizations of untwisted affine Kac-Moody algebras. The well-known Sugawara construction of the Virasoro algebra from the loop algebra realizations of untwisted affine Kac-Moody algebras is presented. Furthermore, we give the well-known "coset construction" introduced by Goddard, Kent and Olive [GKO1, GKO2], which is related to the string functions.

In Chapter 5, we show that the space spanned by the characters of integrable highest-weight irreducible modules of affine Kac-Moody algebras at a given level is invariant under the modular transformations. Moreover, the imaginary root string functions are modular forms. First, we introduce Heisenberg groups and theta functions related to Laurentz integral linear lattices. We define an action of Heisenberg groups on functions and use it to characterize these theta functions. Then we discuss the modular transformation properties of the theta functions. Moreover, we show that certain specializations of the theta functions are modular forms. A generalization of the strange formula found by Freudenthal and de Vries is proved. Finally, we apply the theory of theta functions to affine Kac-Moody algebras.

In Chapter 6, we use generating functions and bosonic, fermionic fields to realize some integrable highest-weight irreducible representations of affine Kac-Moody algebras. First, we present preliminaries in calculus of formal variables, and generating functions for affine Kac-Moody algebras and the Virasoro algebra. Then vertex operator representations of affine Kac-Moody algebras are given. Moreover, we give free fermionic field realizations of certain modules of the general linear and orthogonal affine algebras. The well-known "Boson-Fermion correspondence" in physics is given, and its connection with integrable systems is presented. As an application, we give the vertex operator realizations of the level-one representations of the affine algebra of type $B_\ell^{(1)}$.

Finally, we list some important open problems in the theory of Kac-Moody algebras:

(1) Find the generating functions of the imaginary root multiplicities of Kac-Moody algebras of indefinite type.

(2) Find the singular vectors (or highest-weight vectors) in the Verma modules of Kac-Moody algebras, where the highest weight may not be dom-

inant integral.

(3) Find the string functions of integrable highest-weight modules of affine Kac-Moody algebras.

(4) Find asymptotic formula for the q-dimensions of integrable highest-weight modules of Kac-Moody algebras and related partition functions such as Kostant functions.

Chapter 1

Structure of Kac-Moody Algebras

In this chapter, we give the basic settings of Kac-Moody algebras and study their structures. In Section 1.1, we introduce the Lie algebra $\mathcal{G}(A)$ associated with any square matrix without zero rows and columns. This more general setting will be used later to prove the Chevalley's relations on generators of Kac-Moody algebras. A nondegenerate invariant bilinear form of the Lie algebra $\mathcal{G}(A)$ is constructed in Section 1.2 when A is symmetrizable. Moreover, we introduce a generalized Casimir operator in Section 1.3. The Weyl group for a Kac-Moody algebra is introduced in Section 1.4. Section 1.5 is devoted to a certain classification of generalized Cartan matrices. Finally in Section 1.6, real and imaginary roots are defined and characterized.

1.1 Lie Algebra Associated with a Matrix

In this section, we will introduce the Lie algebra associated with any square matrix and study its basic properties.

Let V be a vector space and let V^* be its dual (the space of linear functions on V). Throughout this book, we will use the notation:

$$\langle \lambda, u \rangle = \langle u, \lambda \rangle = \lambda(u) \qquad \text{for } \lambda \in V^*,\ u \in V \tag{1.1.1}$$

and the index set

$$\overline{i,j} = \{i, i+1, \cdots, j\} \tag{1.1.2}$$

for any two integers i, j with $i \leqslant j$.

Suppose that $A = (a_{i,j})_{n\times n}$ is an $n \times n$ matrix of rank ℓ without zero rows and columns. After re-indexing if necessary, we assume that first ℓ rows are linearly independent. Denote by I_k the $k \times k$ identity matrix. Write

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \tag{1.1.3}$$

where A_1 is an $\ell \times n$ matrix of rank ℓ. Consider

$$C = \begin{pmatrix} A_1 & 0 \\ A_2 & I_{n-\ell} \end{pmatrix}. \tag{1.1.4}$$

Take

$$H = \mathbb{C}^{2n-\ell}, \quad \text{the row space,} \tag{1.1.5}$$

$\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ to be the first n coordinate functions of H and $\Pi^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \cdots, \alpha_n^\vee\}$ to be rows of C. Then both Π and $\Pi^\vee$ are linearly independent, and

$$a_{i,j} = \langle \alpha_i^\vee, \alpha_j \rangle \qquad \text{for } i, j \in \overline{1,n}. \tag{1.1.6}$$

The triple $(H, \Pi, \Pi^\vee)$ is called a *realization* of A.

Set

$$Q = \sum_{i=1}^n \mathbb{Z}\alpha_i, \quad Q_+ = \sum_{i=1}^n \mathbb{Z}_+\alpha_i. \tag{1.1.7}$$

Denote

$$\mathrm{ht}\,\alpha = \sum_{i=1}^n k_i \qquad \text{for } \alpha = \sum_{i=1}^n k_i\alpha_i \in Q. \tag{1.1.8}$$

Moreover, we define a partial ordering $\geqslant$ on Q by

$$\alpha \leqslant \beta \qquad \text{if } \beta - \alpha \in Q_+. \tag{1.1.9}$$

Define $\tilde{\mathcal{G}}(A)$ to be a Lie algebra generated by $\{e_i, f_i \mid i \in \overline{1,n}\}$ and H with the defining relations:

$$[e_i, f_j] = \delta_{i,j}\alpha_j^\vee, \qquad [h, h'] = 0, \tag{1.1.10}$$

$$[h, e_i] = \langle \alpha_i, h\rangle e_i, \qquad [h, f_i] = -\langle \alpha_i, h\rangle f_i \tag{1.1.11}$$

for $i, j \in \overline{1,n}$ and $h, h' \in H$. Let V be the tensor algebra on a vector space with a basis $\{v_1, v_2, \cdots, v_n\}$. For convenience, we simply denote

$$v_{i_1} \otimes \cdots \otimes v_{i_r} = v_{i_1} \cdots v_{i_r}. \tag{1.1.12}$$

Given $\lambda \in H^*$, we define an action of $\tilde{\mathcal{G}}(A)$ on V by

$$h(1) = \langle \lambda, h\rangle 1, \quad h(v_{i_1} \cdots v_{i_r}) = \Big(\langle \lambda, h\rangle - \sum_{s=1}^r \langle \alpha_{i_s}, h\rangle\Big) v_{i_1} \cdots v_{i_r}, \tag{1.1.13}$$

$f_j(u) = v_j u$, $e_j(1) = 0$, $e_j(v_i) = \delta_{i,j}\langle \lambda, \alpha_i^\vee\rangle$ and

$$e_j(v_{i_1} \cdots v_{i_r}) = v_{i_1} e_j(v_{i_2} \cdots v_{i_r}) + \delta_{i_1,j}\Big(\langle \lambda, \alpha_j^\vee\rangle - \sum_{s=2}^r a_{j,i_s}\Big) v_{i_2} \cdots v_{i_r} \tag{1.1.14}$$

for $h \in H$, $u \in V$ and $j \in \overline{1,n}$.