Matrix Methods and Fractional Calculus

Arak M Mathai
Hans J Haubold

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Fractional calculus in terms of mathematics and statistics and its applications to problems in natural sciences is NOT yet part of university teaching curricula. This book is one attempt to provide an approach to include topics of fractional calculus into university curricula. Additionally the material is useful for people who do research work in the areas of special functions, fractional calculus, applications of fractional calculus, and mathematical statistics.

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Preface

This book is an abridged version of the lectures given at the 2014 and 2015 SERB Schools at CMSS. For the 2014 and 2015 SERB Schools the topic was Matrix Methods and Fractional Calculus. Foreign lecturers included Professor Francesco Mainardi from Italy and Professor Serge B. Provost from Canada. Professor Rudolf Gorenflo from Germany and Professor Hans J. Haubold from Austria and the United Nations were supposed to come to give lectures but due to unexpected health problems they could not make it. Indian lecturers included Professors V. Daftardar-Gejji from Pune, M.A. Pathan from CMSS and Aligarh, R.B. Bapat from New Delhi and N. Mukunda from Bengaluru. The course Director is Professor A.M. Mathai (India/Canada) who is also the Director of CMSS. Professor N. Mukunda lectured on the applications of Hermitian positive definite matrices in quantum mechanics, light scattering and other areas and Professor R.B. Bapat lectured on matrix methods in graph theory. These two lecturers did not make available their lecture notes and hence they are not included in the current book. Professor A.M. Mathai lectured on mathematical and statistical preliminaries, matrix-variate statistical distributions, functions of matrix argument etc. Professor Mainardi lectured on the analysis aspects of fractional calculus and Professor Daftardar-Gejji lectured on the applications of fractional differential equations in control theory and engineering problems. Professor Pathan's lectures were on Lie Groups, Lie Algebra connected with special functions and Professor S.B. Provost from the University of Western Ontario, Canada, lectured on some aspects of multivariate statistical analysis.

Chapters 1 and 2 provide the mathematical and statistical preliminaries, vector and matrix differential operators, their applications in quadratic and bilinear forms, maxima/minima problems, optimizations, Jacobians of matrix transformations and functions of matrix argument. Chapter 3 is on the theory of fractional integrals and fractional derivatives, Mittag-Leffler functions and their properties. Mittag-Leffler function is considered as the queen function in fractional calculus. Chapter 4 gives fractional differential equations, applications of fractional calculus in engineering and control theory problems, Adomian decomposition and iterative methods for the solutions of fractional differential equations. Chapter 5 is on the recent developments on matrix-variate fractional integrals and fractional derivatives or fractional calculus for functions of matrix argument. Chapter 6 is on Lie theory and special functions. Chapter 7 is on some aspects of multivariate statistical analysis, multivariate Gaussian and Wishart densities, their properties, tests of statistical hypotheses etc.

The material is useful for people who do research work in the areas of special functions, fractional calculus, applications of fractional calculus, and mathematical statistics, especially the multivariate and matrix-variate statistical distributions, tests of hypotheses, etc. Since the material is based on lecture notes, the material is also good for someone to get into these areas for their research work.

Dr H.J. Haubold was an integral part of these SERC Schools, one of the organizers and one of the foreign lecturers. Then A.M. Mathai and H.J. Haubold organized a national level conference on fractional calculus in 2012 at CMSS Pala Campus. Then the lecture notes were updated and brought out as Module 10 of CMSS in August 2014. A second printing of Module 10 took place in 2015.

The 2014 SERB Notes were brought out in the Publications Series of CMSS as Publication Number 44 of CMSS. Publications Series of CMSS consist of research level books and monographs. Since there was much overlap with the material of 2014 SERB School, no separate publication was brought out for the 2015 SERB

Notes. Research level Publication Number 44 of CMSS was developed with financial support from DST, Government of India, New Delhi, under Project Number SR/S4/MS:783/12. The authors would like to express their sincere gratitude to DST, Government of India, New Delhi, for the financial assistance.

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2nd April 2017

Acknowledgments

This is a modified version of CMSS (Centre for Mathematical and Statistical Sciences) 2014 and 2015 SERB School Notes. SERB Schools (Science and Engineering Research Board of the Department of Science and Technology, Government of India) are annual four weeks intensive all-India research level course on the topic. CMSS (formerly CMS = Centre for Mathematical Sciences) has been conducting these Schools from 1995 onward. In the earlier years the Schools used to be of six weeks duration but from 2005 onward the duration was cut down to four weeks. The first sequence of Schools was on Special Functions and Their Applications. Summarized version of these notes was brought out as the Springer, New York, publication *Special Functions for Applied Scientists* in 2008. The second sequence was on Functions of Matrix Argument and Their Applications. The third sequence was on Matrix Methods and Fractional Calculus. From 2006 onward, Fractional Calculus became an integral part of the SERC (Science and Engineering Research Council, later became research board or SERB) Schools at CMSS.

These CMSS SERB Notes are printed at CMSS Press and published by CMSS. Copies are made available to students free of cost and to researchers and others at production cost. For the preparation and printing of these publications, financial assistance was available from the Department of Science and Technology, Government of India (DST), New Delhi under project number SR/S4/MS:783/12.

Hence the authors would like to express their thanks and gratitude to DST, Government of India, for the financial assistance.

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2nd April 2017

List of Symbols

$\lvert(\cdot)\rvert$	absolute value/determinant of $(\cdot)$/Section 1.1, p. 1
$\mathrm{tr}(X)$	trace of the matrix X/Section 1.1, p. 1
$\mathrm{d}X$	wedge product of differentials/Section 1.1, p. 2
J	Jacobian/Section 1.1, p. 3
$X > O, X \geq O$	definiteness of matrices/Section 1.1, p. 6
$X < O, X \leq O$	definiteness of matrices/Section 1.1, p. 6
$\int_A^B f(X)\mathrm{d}X$	integral over all X such that/Section 1.1, p. 6 $X > O, X - A > O, B - X > O$/Section 1.1, p. 6
X^*	conjugate transpose of X/Section 1.1, p. 6
$\frac{\partial}{\partial X}$	vector differential operator/Section 1.2, p. 7
X'	transpose the matrix X/Section 1.2, p. 7
$\mathrm{diag}(a_{11}, \ldots, a_{pp})$	diagonal matrix/Section 1.4, p. 41
$\otimes$	Kronecker product/Section 2.3, p. 71
$\Gamma_p(\alpha)$	real matrix-variate gamma/Section 2.4, p. 74
$B_p(\alpha, \beta)$	real matrix-variate beta/Section 2.4, p. 76
$(\mathrm{d}X)$	matrix of differentials $\mathrm{d}x_{ij}$'s/Section 2.5, p. 86
$\tilde{\Gamma}_p(\alpha)$	complex matrix-variate gamma/Eq. (2.6.7), p. 97
I_{a+}^{α}	left-sided fractional integral/Eq. (3.2.3), p. 107
I_{b-}^{α}	right-sided fractional integral/Eq. (3.2.5), p. 108

D^n	integer-order derivative/Section 3.2, p. 109
D^{α}_{a+}	left-sided fractional derivative/Section 3.2, p. 110
D^{α}_{b-}	right-sided fractional derivative/Section 3.2, p. 110
$E_{\alpha}(z)$	Mittag-Leffler functions/Eq. (3.6.1), p. 129
$E_{\alpha,\beta}(z)$	Mittag-Leffler function/Eq. (3.6.5), p. 130
$\mathrm{erf}(z)$	error function/Eq. (3.6.4), p. 129
$\Phi_{\lambda,\mu}(z)$	Wright function/Eq. (3.7.1), p. 143
$J^{(\lambda)}_{\nu}(z)$	Wright generalized Bessel function/Eq. (3.7.3), p. 145
${}^{c}D^{\alpha}$	Caputo fractional derivative/Eq. (3.5.11), p. 124
$K^{-\alpha}_{2,u,\gamma}$	Kober integral, second kind/Eq. (5.3.2), p. 207
$K^{-\alpha}_{1,u,\gamma}$	Kober integral, first kind/Eq. (5.4.1), p. 209
$C_K(Z)$	zonal polynomial/Section 5.4, p. 211
$(a)_K$	generalized Pochhmmer symbol/Eq. (5.4.7), p. 211
$\tilde{D}^{\alpha}_{1}, \tilde{D}^{\alpha}_{2}$	fractional derivatives, matrix-variate/ Section 5.5, p. 214
e^{A}	matrix series/Definition 6.2.1, p. 220
$\mathrm{GL}(2, C)$	general linear group/Example 6.5.3, p. 232
j^{+}, j^{-}	Lie group elements/Example 6.5.3, p. 232
$f_{x_1,\ldots,x_k}(X)$	multivariate density/Section 7.2.1, p. 248
$F_{x_1,\ldots,x_k}(X)$	multivariate distribution function/Eq. (7.2.1), p. 248
$E[g(X)]$	expected value/Section 7.2.1, p. 250
$N_p(\mu, V)$	multivariate normal density/Section 7.2.3, p. 253

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