美国数学会经典影印系列
AMS
AMERICAN MATHEMATICAL SOCIETY

Introduction to the *h*-Principle

h-原理引论

Y. Eliashberg, N. Mishachev

高等教育出版社

美国数学会经典影印系列
AMS
AMERICAN MATHEMATICAL SOCIETY

Introduction to the *h*-Principle

h-原理引论

Y. Eliashberg, N. Mishachev

高等教育出版社·北京

图字：01-2016-2503 号

h-原理引论
h-Yuanli Yinlun

图书在版编目 (CIP) 数据

h-原理引论 = Introduction to the h-Principle：英文 / (美) 伊利埃伯格 (Y. Eliashberg)，(俄罗斯) 米哈切夫 (N. Mishachev) 著．—影印本．—北京：高等教育出版社，2017.1
ISBN 978-7-04-046902-8
Ⅰ．① h… Ⅱ．①伊… ②米… Ⅲ．①微分几何—英文 ②微分流形—英文 ③微分方程—英文 Ⅳ．① O186.1 ② O189.3 ③ O175
中国版本图书馆 CIP 数据核字 (2016) 第 282281 号

策划编辑 赵天夫　　责任编辑 赵天夫
封面设计 张申申　　责任印制 毛斯璐

出版发行 高等教育出版社
社址 北京市西城区德外大街 4 号
邮政编码 100120
购书热线 010-58581118
咨询电话 400-810-0598
网址 http://www.hep.edu.cn
http://www.hep.com.cn
网上订购 http://www.hepmall.com.cn
http://www.hepmall.com
http://www.hepmall.cn
印刷:北京新华印刷有限公司

开本 787mm × 1092mm 1/16
印张 14.25
字数 320 千字
版次 2017 年 1 月第 1 版
印次 2017 年 1 月第 1 次印刷
定价 99.00 元

[物 料 号 46902-00]

美国数学会经典影印系列

出版者的话

近年来，我国的科学技术取得了长足进步，特别是在数学等自然科学基础领域不断涌现出一流的研究成果。与此同时，国内的科研队伍与国外的交流合作也越来越密切，越来越多的科研工作者可以熟练地阅读英文文献，并在国际顶级期刊发表英文学术文章，在国外出版社出版英文学术著作。

然而，在国内阅读海外原版英文图书仍不是非常便捷。一方面，这些原版图书主要集中在科技、教育比较发达的大中城市的大型综合图书馆以及科研院所的资料室中，普通读者借阅不甚容易；另一方面，原版书价格昂贵，动辄上百美元，购买也很不方便。这极大地限制了科技工作者对于国外先进科学技术知识的获取，间接阻碍了我国科技的发展。

高等教育出版社本着植根教育、弘扬学术的宗旨服务我国广大科技和教育工作者，同美国数学会（American Mathematical Society）合作，在征求海内外众多专家学者意见的基础上，精选该学会近年出版的数十种专业著作，组织出版了“美国数学会经典影印系列”丛书。美国数学会创建于1888年，是国际上极具影响力的专业学术组织，目前拥有近30000会员和580余个机构成员，出版图书3500多种，冯·诺依曼、莱夫谢茨、陶哲轩等世界级数学大家都是其作者。本影印系列涵盖了代数、几何、分析、方程、拓扑、概率、动力系统等所有主要数学分支以及新近发展的数学主题。

我们希望这套书的出版，能够对国内的科研工作者、教育工作者以及青年学生起到重要的学术引领作用，也希望今后能有更多的海外优秀英文著作被介绍到中国。

高等教育出版社

2016年12月

To

Vladimir Igorevich Arnold

who introduced us to the world of singularities

and

Misha Gromov

who taught us how to get rid of them

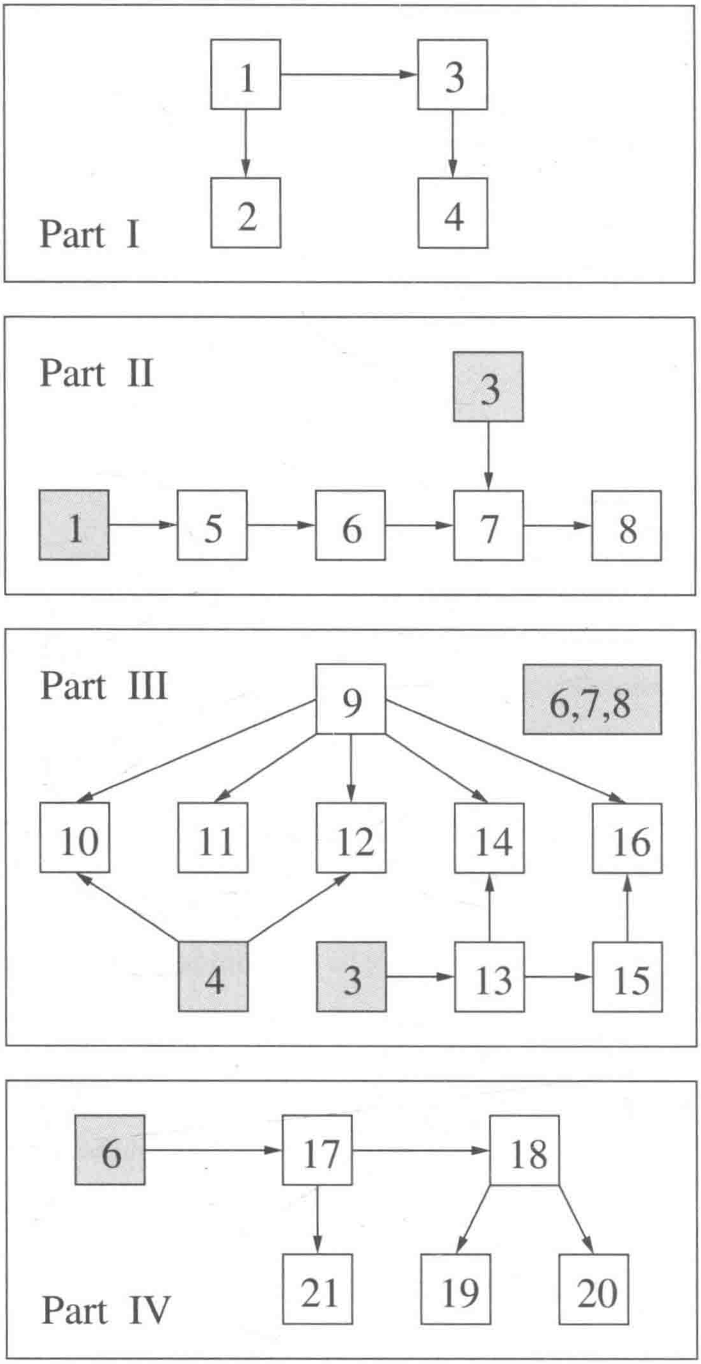

Figure 0.1. The relations between chapters of the book

Preface

A *partial differential relation* $\mathcal{R}$ is any condition imposed on the partial derivatives of an unknown function. A solution of $\mathcal{R}$ is any function which satisfies this relation.

The classical partial differential relations, mostly rooted in Physics, are usually described by (systems of) equations. Moreover, the corresponding systems of equations are mostly *determined*: the number of unknown functions is equal to the number of equations. Given appropriate boundary conditions, such a differential relation usually has a unique solution. In some cases this solution can be found using certain *analytical* methods (potential theory, Fourier method and so on).

In differential geometry and topology one often deals with systems of partial differential equations, as well as partial differential *inequalities*, which have infinitely many solutions whatever boundary conditions are imposed. Moreover, sometimes solutions of these differential relations are C^0-dense in the corresponding space of functions or mappings. The systems of differential equations in question are usually (but not necessarily) *underdetermined*. We discuss in this book *homotopical* methods for solving this kind of differential relations. Any differential relation has an underlying algebraic relation which one gets by substituting derivatives by new independent variables. A solution of the corresponding algebraic relation is called a *formal* solution of the original differential relation $\mathcal{R}$. Its existence is a necessary condition for the solvability of $\mathcal{R}$, and it is a natural starting point for exploring $\mathcal{R}$. Then one can try to deform the formal solution into a genuine solution. We say that the *h-principle* holds for a differential relation $\mathcal{R}$ if any formal solution of $\mathcal{R}$ can be deformed into a genuine solution.

The *notion* of h-principle (under the name "w.h.e.-principle") first appeared in [**Gr71**] and [**GE71**]. The *term* "h-principle" was introduced and popularized by M. Gromov in his book [**Gr86**]. The h-principle for solutions of partial differential relations exposed the soft/hard (or flexible/rigid) dichotomy for the problems formulated in terms of derivatives: a particular analytical problem is "soft" or "abides by the h-principle" if its solvability is determined by some underlying *algebraic* or *geometric* data. The softness phenomena was first discovered in the fifties by J. Nash [**Na54**] for isometric C^1-immersions, and by S. Smale [**Sm58, Sm59**] for differential immersions. However, instances of soft problems appeared earlier (e.g. H. Whitney's paper [**Wh37**]). In the sixties several new geometrically interesting examples of soft problems were discovered by M. Hirsch, V. Poénaru, A. Phillips, S. Feit and other authors (see [**Hi59**], [**Po66**], [**Ph67**], [**Fe69**]). In his dissertation [**Gr69**], in the paper [**Gr73**] and later in his book [**Gr86**], Gromov transformed Smale's and Nash's ideas into two powerful general methods for solving partial differential relations: *continuous sheaves* (or the *covering homotopy*) method and the *convex integration* method. The third method, called *removal of singularities*, was first introduced and explored in [**GE71**].

There is an opinion that *"the h-principle is the hardest part of Gromov's work to popularize"* (see [**Be00**]). We have written our book in order to improve the situation. We consider here two geometrical methods: *holonomic approximation*, which is a version of the method of *continuous sheaves*, and *convex integration*. We do not pretend to cover here the content of Gromov's book [**Gr86**], but rather want to prepare and motivate the reader to look for hidden treasures there. On the other hand, the reader interested in applications will find that with a few notable exceptions (e.g. Lohkamp's theory [**Lo95**] of negative Ricci curvature and Donaldson's theory [**Do96**] of approximately holomorphic sections) most instances of the h-principle which are known today can be treated by the methods considered in the present book.

The first three parts of the book are devoted to a quite general theorem about holonomic approximation of sections of jet-bundles and its applications. Given an arbitrary submanifold $V_0 \subset V$ of positive codimension, the Holonomic Approximation Theorem allows us to solve any *open* differential relations $\mathcal{R}$ near a slightly perturbed submanifold $\widetilde{V}_0 = h(V)$ where $h : V \to V$ is a C^0-small diffeomorphism. Gromov's h-principle for open Diff V-invariant differential relations on open manifolds, his directed embedding theorem, as well as some other results in the spirit of the h-principle are immediate corollaries of the Holonomic Approximation Theorem.

The method for proving the h-principle based on the Holonomic Approximation Theorem works well for *open* manifolds. Applications to closed manifolds require an additional trick, called *microextension*. It was first used by M. Hirsch in [**Hi59**]. The holonomic approximation method also works well for differential relations which are not open, but *microflexible*. The most interesting applications of this type come from Symplectic Geometry. These applications are discussed in the third part of the book. For convenience of the reader the basic notions of Symplectic Geometry are also reviewed in that part of the book.

The fourth part of the book is devoted to *convex integration theory*. Gromov's convex integration theory was treated in great detail by D. Spring in [**Sp98**]. In our exposition of convex integration we pursue a different goal. Rather than considering the sophisticated advanced version of convex integration presented in [**Gr86**], we explore only its simple version for first order differential relations, similar to the first exposition of the theory by Gromov in [**Gr73**]. Nevertheless, we prove here practically all the most interesting corollaries of the theory, including the Nash-Kuiper theorem on C^1-isometric embeddings.

Let us list here some available books and survey papers about the h-principle. Besides Gromov's book [**Gr86**], these are: Spring's book [**Sp98**], Adachi's book [**Ad93**], Haefliger's paper [**Ha71**], Poénaru's paper [**Po71**] and, most recently, Geiges' notes [**Ge01**].

Acknowledgements. The book was partially written while the second author visited the Department of Mathematics of Stanford University, and the first author visited the Mathematical Institute of Leiden University and the Institute for Advanced Study at Princeton. The authors thank the host institutions for their hospitality. While writing this book the authors were partially supported by the National Science Foundation. The first author also acknowledges the support of The Veblen Fund during his stay at the IAS.

We are indebted to Ana Cannas da Silva, Hansjorg Geiges, Simon Goberstein, Dusa McDuff and David Spring who read the preliminary version of this book and corrected numerous misprints and mistakes. We are very thankful to all the mathematicians who communicated to us their critical remarks and suggestions.

Contents

Intrigue

◂ **Examples**

A. Immersions. A smooth map $f : V \to W$ of an n-dimensional manifold V into a q-dimensional manifold W, $n \leq q$, is called an *immersion* if its differential has the maximal rank n at every point. Two immersions are called *regularly homotopic* if one can be deformed to the other through a smooth family of immersions.

A1. For an immersion $f : S^1 \to \mathbb{R}^2$ we denote by $G(f)$ its *tangential degree*, i.e. the degree of the corresponding Gaussian map $S^1 \to S^1$. Then *two immersions $f, g : S^1 \to \mathbb{R}^2$ are regularly homotopic if and only if $G(f) = G(g)$*, see [**Wh37**] and Section 6.1 below.

A2. On the other hand, *any two immersions $S^2 \to \mathbb{R}^3$ are regularly homotopic*, see [**Sm58**] and Section 4.2 below. In particular, the standard 2-sphere in $\mathbb{R}^3$ can be inverted outside in through a family of immersions.

A3. Consider now pairs of immersions $(f_0, f_1) : D^2 \to \mathbb{R}^2$ which coincide near the boundary circle ∂D^2. What is the classification of such pairs up to the regular homotopy in this class? The answer turns out to be quite unexpected:

There are precisely two regular homotopy classes of such pairs. One is represented by the pair (j, j) where j is the inclusion $D^2 \hookrightarrow \mathbb{R}^2$, the second one is represented by the pair (f, g) where the immersions f and g are shown in Fig. 0.2. See [**El72**].

B. Isometric C^1-immersions. Is there a regular homotopy $f_t : S^2 \to \mathbb{R}^3$ which begins with the inclusion f_0 of the unit sphere and ends with an isometric immersion f_1 into the ball of radius $\frac{1}{2}$? Here the word 'isometric' means *preserving length of all curves.* The answer is, of course, negative if f_1 is required to be C^2-smooth. Indeed, in this case the Gaussian curvature of

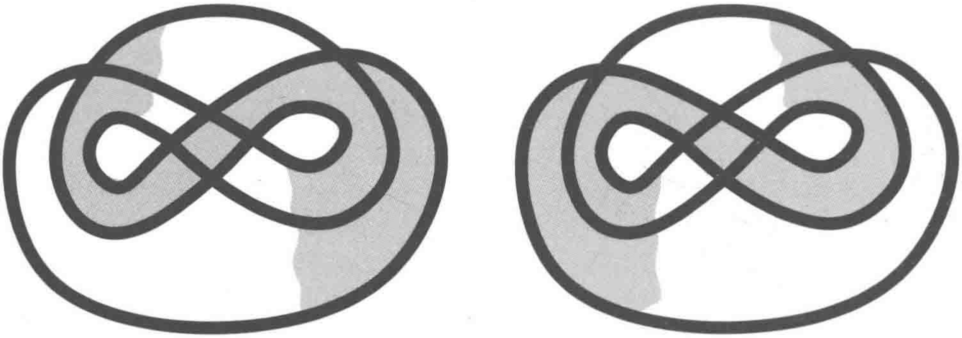

Figure 0.2. The immersions f and g.

the metric on S^2 should be ≥ 4 at least somewhere. However, surprisingly, *the answer is "yes" in the case of* C^1*-immersions* (when the curvature is not defined but the curve length is), see [**Na54, Ku55**] and Chapter 21 below.

C. Mappings with a prescribed Jacobian. *Let* Ω *be an* n*-form on a closed oriented stably parallelizable* n*-dimensional manifold* M *such that* $\int_M \Omega = 0$*, and let*

$$\eta = dx_1 \wedge \cdots \wedge dx_n$$

be the standard volume form on $\mathbb{R}^n$*. Then there exists a map* $f : M \to \mathbb{R}^n$ *such that* $f^*\eta = \Omega$. See [**GE73**]. ►

All the above statements are examples of the *homotopy principle*, or the *h-principle.* Despite the fact that all these problems are asking for the solution of certain differential equations or inequalities, they can be reduced to problems of a pure homotopy-theoretic nature which then can be dealt with using the methods of Algebraic topology. For instance, the regular homotopy classification of immersions $S^2 \to \mathbb{R}^3$ can be reduced to the computation of the homotopy group $\pi_2(\mathbb{R}P^3)$, which is trivial.

We are teaching in this book how to deal with these problems. In particular, two general methods which we describe here will be sufficient to handle all the above examples, except **A3** and **C**. In our sequel book, "The h-Principle and Singularities", we will discuss other methods which prove, in particular, the two remaining results.

Another, maybe even more important, goal of this book is to teach the reader *how to recognize* the problems which may satisfy the h-principle. Of course, in the most interesting cases this is a very difficult question. As we will see below there are plenty of open problems where one neither can establish the h-principle, nor find a single instance of *rigidity*. Nevertheless