

美国数学会经典影印系列



# Introduction to the $h$ -Principle

$h$ -原理引论

Y. Eliashberg, N. Mishachev



高等教育出版社

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## 出版者的话

近年来,我国的科学技术取得了长足进步,特别是在数学等自然科学基础领域不断涌现出一流的研究成果。与此同时,国内的科研队伍与国外的交流合作也越来越密切,越来越多的科研工作者可以熟练地阅读英文文献,并在国际顶级期刊发表英文学术文章,在国外出版社出版英文学术著作。

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我们希望这套书的出版,能够对国内的科研工作者、教育工作者以及青年学生起到重要的学术引领作用,也希望今后能有更多的海外优秀英文著作被介绍到中国。

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2016年12月

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*To*

**Vladimir Igorevich Arnold**

*who introduced us to the world of singularities*

*and*

**Misha Gromov**

*who taught us how to get rid of them*

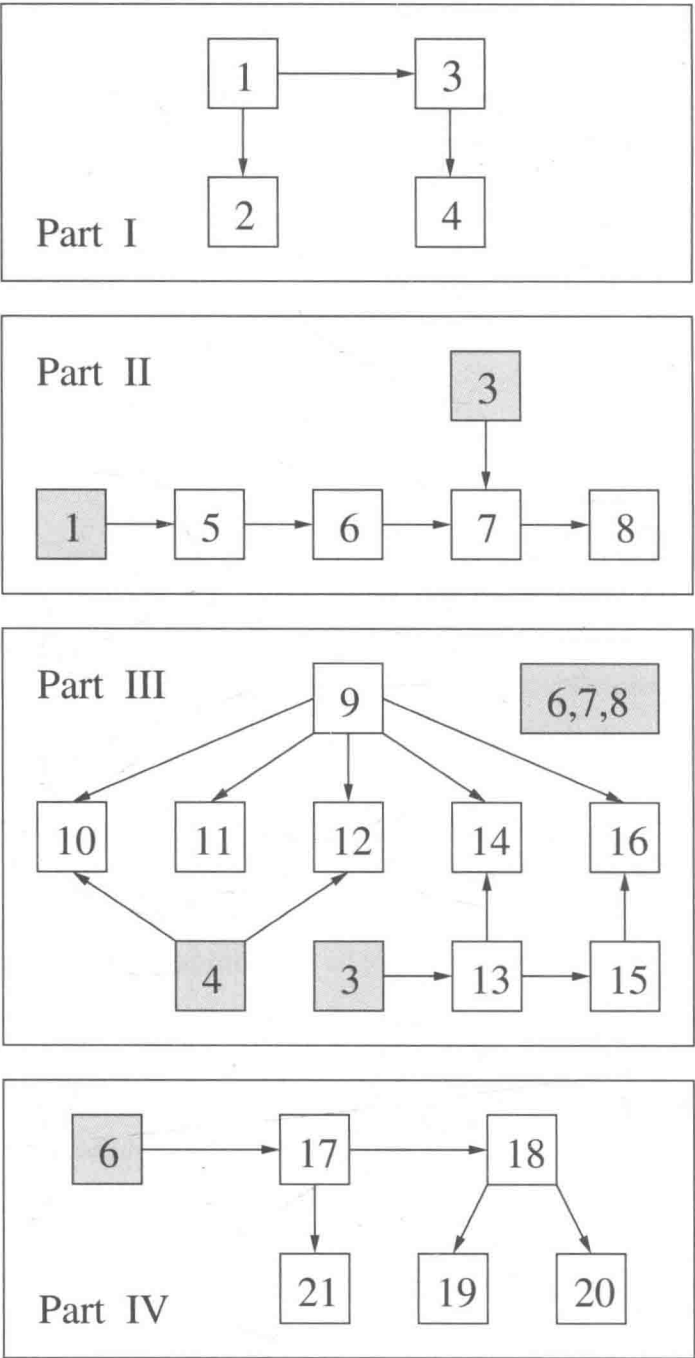


Figure 0.1. The relations between chapters of the book

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# Preface

A *partial differential relation*  $\mathcal{R}$  is any condition imposed on the partial derivatives of an unknown function. A solution of  $\mathcal{R}$  is any function which satisfies this relation.

The classical partial differential relations, mostly rooted in Physics, are usually described by (systems of) equations. Moreover, the corresponding systems of equations are mostly *determined*: the number of unknown functions is equal to the number of equations. Given appropriate boundary conditions, such a differential relation usually has a unique solution. In some cases this solution can be found using certain *analytical* methods (potential theory, Fourier method and so on).

In differential geometry and topology one often deals with systems of partial differential equations, as well as partial differential *inequalities*, which have infinitely many solutions whatever boundary conditions are imposed. Moreover, sometimes solutions of these differential relations are  $C^0$ -dense in the corresponding space of functions or mappings. The systems of differential equations in question are usually (but not necessarily) *underdetermined*. We discuss in this book *homotopical* methods for solving this kind of differential relations. Any differential relation has an underlying algebraic relation which one gets by substituting derivatives by new independent variables. A solution of the corresponding algebraic relation is called a *formal* solution of the original differential relation  $\mathcal{R}$ . Its existence is a necessary condition for the solvability of  $\mathcal{R}$ , and it is a natural starting point for exploring  $\mathcal{R}$ . Then one can try to deform the formal solution into a genuine solution. We say that the *h-principle* holds for a differential relation  $\mathcal{R}$  if any formal solution of  $\mathcal{R}$  can be deformed into a genuine solution.



The *notion* of  $h$ -principle (under the name “w.h.e.-principle”) first appeared in [Gr71] and [GE71]. The *term* “ $h$ -principle” was introduced and popularized by M. Gromov in his book [Gr86]. The  $h$ -principle for solutions of partial differential relations exposed the soft/hard (or flexible/rigid) dichotomy for the problems formulated in terms of derivatives: a particular analytical problem is “soft” or “abides by the  $h$ -principle” if its solvability is determined by some underlying *algebraic* or *geometric* data. The softness phenomena was first discovered in the fifties by J. Nash [Na54] for isometric  $C^1$ -immersions, and by S. Smale [Sm58, Sm59] for differential immersions. However, instances of soft problems appeared earlier (e.g. H. Whitney’s paper [Wh37]). In the sixties several new geometrically interesting examples of soft problems were discovered by M. Hirsch, V. Poénaru, A. Phillips, S. Feit and other authors (see [Hi59], [Po66], [Ph67], [Fe69]). In his dissertation [Gr69], in the paper [Gr73] and later in his book [Gr86], Gromov transformed Smale’s and Nash’s ideas into two powerful general methods for solving partial differential relations: *continuous sheaves* (or the *covering homotopy*) method and the *convex integration* method. The third method, called *removal of singularities*, was first introduced and explored in [GE71].

There is an opinion that “*the  $h$ -principle is the hardest part of Gromov’s work to popularize*” (see [Be00]). We have written our book in order to improve the situation. We consider here two geometrical methods: *holonomic approximation*, which is a version of the method of *continuous sheaves*, and *convex integration*. We do not pretend to cover here the content of Gromov’s book [Gr86], but rather want to prepare and motivate the reader to look for hidden treasures there. On the other hand, the reader interested in applications will find that with a few notable exceptions (e.g. Lohkamp’s theory [Lo95] of negative Ricci curvature and Donaldson’s theory [Do96] of approximately holomorphic sections) most instances of the  $h$ -principle which are known today can be treated by the methods considered in the present book.

The first three parts of the book are devoted to a quite general theorem about holonomic approximation of sections of jet-bundles and its applications. Given an arbitrary submanifold  $V_0 \subset V$  of positive codimension, the Holonomic Approximation Theorem allows us to solve any *open* differential relations  $\mathcal{R}$  near a slightly perturbed submanifold  $\tilde{V}_0 = h(V)$  where  $h : V \rightarrow V$  is a  $C^0$ -small diffeomorphism. Gromov’s  $h$ -principle for open Diff  $V$ -invariant differential relations on open manifolds, his directed embedding theorem, as well as some other results in the spirit of the  $h$ -principle are immediate corollaries of the Holonomic Approximation Theorem.

The method for proving the  $h$ -principle based on the Holonomic Approximation Theorem works well for *open* manifolds. Applications to closed manifolds require an additional trick, called *microextension*. It was first used by M. Hirsch in [Hi59]. The holonomic approximation method also works well for differential relations which are not open, but *microflexible*. The most interesting applications of this type come from Symplectic Geometry. These applications are discussed in the third part of the book. For convenience of the reader the basic notions of Symplectic Geometry are also reviewed in that part of the book.

The fourth part of the book is devoted to *convex integration theory*. Gromov's convex integration theory was treated in great detail by D. Spring in [Sp98]. In our exposition of convex integration we pursue a different goal. Rather than considering the sophisticated advanced version of convex integration presented in [Gr86], we explore only its simple version for first order differential relations, similar to the first exposition of the theory by Gromov in [Gr73]. Nevertheless, we prove here practically all the most interesting corollaries of the theory, including the Nash-Kuiper theorem on  $C^1$ -isometric embeddings.

Let us list here some available books and survey papers about the  $h$ -principle. Besides Gromov's book [Gr86], these are: Spring's book [Sp98], Adachi's book [Ad93], Haefliger's paper [Ha71], Poénaru's paper [Po71] and, most recently, Geiges' notes [Ge01].

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# Contents

Preface	xv
Intrigue	1
<b>Part 1. Holonomic Approximation</b>	
Chapter 1. Jets and Holonomy	7
§1.1. Maps and sections	7
§1.2. Coordinate definition of jets	7
§1.3. Invariant definition of jets	9
§1.4. The space $X^{(1)}$	10
§1.5. Holonomic sections of the jet space $X^{(r)}$	11
§1.6. Geometric representation of sections of $X^{(r)}$	12
§1.7. Holonomic splitting	12
Chapter 2. Thom Transversality Theorem	15
§2.1. Generic properties and transversality	15
§2.2. Stratified sets and polyhedra	16
§2.3. Thom Transversality Theorem	17
Chapter 3. Holonomic Approximation	21
§3.1. Main theorem	21
§3.2. Holonomic approximation over a cube	23
§3.3. Fiberwise holonomic sections	24
§3.4. Inductive Lemma	25

§3.5. Proof of the Inductive Lemma	28
§3.6. Holonomic approximation over a cube	33
§3.7. Parametric case	34
Chapter 4. Applications	37
§4.1. Functions without critical points	37
§4.2. Smale's sphere eversion	38
§4.3. Open manifolds	40
§4.4. Approximate integration of tangential homotopies	41
§4.5. Directed embeddings of open manifolds	44
§4.6. Directed embeddings of closed manifolds	45
§4.7. Approximation of differential forms by closed forms	47
<b>Part 2. Differential Relations and Gromov's <math>h</math>-Principle</b>	
Chapter 5. Differential Relations	53
§5.1. What is a differential relation?	53
§5.2. Open and closed differential relations	55
§5.3. Formal and genuine solutions of a differential relation	56
§5.4. Extension problem	56
§5.5. Approximate solutions to systems of differential equations	57
Chapter 6. Homotopy Principle	59
§6.1. Philosophy of the $h$ -principle	59
§6.2. Different flavors of the $h$ -principle	62
Chapter 7. Open Diff $V$ -Invariant Differential Relations	65
§7.1. Diff $V$ -invariant differential relations	65
§7.2. Local $h$ -principle for open Diff $V$ -invariant relations	66
Chapter 8. Applications to Closed Manifolds	69
§8.1. Microextension trick	69
§8.2. Smale-Hirsch $h$ -principle	69
§8.3. Sections transversal to distribution	71
<b>Part 3. The Homotopy Principle in Symplectic Geometry</b>	
Chapter 9. Symplectic and Contact Basics	75
§9.1. Linear symplectic and complex geometries	75
§9.2. Symplectic and complex manifolds	80

§9.3. Symplectic stability	85
§9.4. Contact manifolds	88
§9.5. Contact stability	94
§9.6. Lagrangian and Legendrian submanifolds	95
§9.7. Hamiltonian and contact vector fields	97
Chapter 10. Symplectic and Contact Structures on Open Manifolds	99
§10.1. Classification problem for symplectic and contact structures	99
§10.2. Symplectic structures on open manifolds	100
§10.3. Contact structures on open manifolds	102
§10.4. Two-forms of maximal rank on odd-dimensional manifolds	103
Chapter 11. Symplectic and Contact Structures on Closed Manifolds	105
§11.1. Symplectic structures on closed manifolds	105
§11.2. Contact structures on closed manifolds	107
Chapter 12. Embeddings into Symplectic and Contact Manifolds	111
§12.1. Isosymplectic embeddings	111
§12.2. Equidimensional isosymplectic immersions	118
§12.3. Isocontact embeddings	121
§12.4. Subcritical isotropic embeddings	128
Chapter 13. Microflexibility and Holonomic $\mathcal{R}$ -Approximation	129
§13.1. Local integrability	129
§13.2. Homotopy extension property for formal solutions	131
§13.3. Microflexibility	131
§13.4. Theorem on holonomic $\mathcal{R}$ -approximation	133
§13.5. Local $h$ -principle for microflexible Diff $V$ -invariant relations	133
Chapter 14. First Applications of Microflexibility	135
§14.1. Subcritical isotropic immersions	135
§14.2. Maps transversal to a contact structure	136
Chapter 15. Microflexible $\mathfrak{A}$ -Invariant Differential Relations	139
§15.1. $\mathfrak{A}$ -invariant differential relations	139
§15.2. Local $h$ -principle for microflexible $\mathfrak{A}$ -invariant relations	140
Chapter 16. Further Applications to Symplectic Geometry	143
§16.1. Legendrian and isocontact immersions	143
§16.2. Generalized isocontact immersions	144

§16.3. Lagrangian immersions	146
§16.4. Isosymplectic immersions	147
§16.5. Generalized isosymplectic immersions	149
<b>Part 4. Convex Integration</b>	
Chapter 17. One-Dimensional Convex Integration	153
§17.1. Example	153
§17.2. Convex hulls and ampleness	154
§17.3. Main lemma	155
§17.4. Proof of the main lemma	156
§17.5. Parametric version of the main lemma	161
§17.6. Proof of the parametric version of the main lemma	162
Chapter 18. Homotopy Principle for Ample Differential Relations	167
§18.1. Ampleness in coordinate directions	167
§18.2. Iterated convex integration	168
§18.3. Principal subspaces and ample differential relations in $X^{(1)}$	170
§18.4. Convex integration of ample differential relations	171
Chapter 19. Directed Immersions and Embeddings	173
§19.1. Criterion of ampleness for directed immersions	173
§19.2. Directed immersions into almost symplectic manifolds	174
§19.3. Directed immersions into almost complex manifolds	175
§19.4. Directed embeddings	176
Chapter 20. First Order Linear Differential Operators	179
§20.1. Formal inverse of a linear differential operator	179
§20.2. Homotopy principle for $\mathcal{D}$ -sections	180
§20.3. Non-vanishing $\mathcal{D}$ -sections	181
§20.4. Systems of linearly independent $\mathcal{D}$ -sections	182
§20.5. Two-forms of maximal rank on odd-dimensional manifolds	184
§20.6. One-forms of maximal rank on even-dimensional manifolds	186
Chapter 21. Nash-Kuiper Theorem	189
§21.1. Isometric immersions and short immersions	189
§21.2. Nash-Kuiper theorem	190
§21.3. Decomposition of a metric into a sum of primitive metrics	191
§21.4. Approximation Theorem	191

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§21.5. One-dimensional Approximation Theorem	193
§21.6. Adding a primitive metric	194
§21.7. End of the proof of the approximation theorem	196
§21.8. Proof of the Nash-Kuiper theorem	196
Bibliography	199
Index	203

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# Intrigue

## ◀ Examples

**A. Immersions.** A smooth map  $f : V \rightarrow W$  of an  $n$ -dimensional manifold  $V$  into a  $q$ -dimensional manifold  $W$ ,  $n \leq q$ , is called an *immersion* if its differential has the maximal rank  $n$  at every point. Two immersions are called *regularly homotopic* if one can be deformed to the other through a smooth family of immersions.

**A1.** For an immersion  $f : S^1 \rightarrow \mathbb{R}^2$  we denote by  $G(f)$  its *tangential degree*, i.e. the degree of the corresponding Gaussian map  $S^1 \rightarrow S^1$ . Then *two immersions  $f, g : S^1 \rightarrow \mathbb{R}^2$  are regularly homotopic if and only if  $G(f) = G(g)$* , see [Wh37] and Section 6.1 below.

**A2.** On the other hand, *any two immersions  $S^2 \rightarrow \mathbb{R}^3$  are regularly homotopic*, see [Sm58] and Section 4.2 below. In particular, the standard 2-sphere in  $\mathbb{R}^3$  can be inverted outside in through a family of immersions.

**A3.** Consider now pairs of immersions  $(f_0, f_1) : D^2 \rightarrow \mathbb{R}^2$  which coincide near the boundary circle  $\partial D^2$ . What is the classification of such pairs up to the regular homotopy in this class? The answer turns out to be quite unexpected:

*There are precisely two regular homotopy classes of such pairs. One is represented by the pair  $(j, j)$  where  $j$  is the inclusion  $D^2 \hookrightarrow \mathbb{R}^2$ , the second one is represented by the pair  $(f, g)$  where the immersions  $f$  and  $g$  are shown in Fig. 0.2. See [El72].*

**B. Isometric  $C^1$ -immersions.** Is there a regular homotopy  $f_t : S^2 \rightarrow \mathbb{R}^3$  which begins with the inclusion  $f_0$  of the unit sphere and ends with an isometric immersion  $f_1$  into the ball of radius  $\frac{1}{2}$ ? Here the word ‘isometric’ means *preserving length of all curves*. The answer is, of course, negative if  $f_1$  is required to be  $C^2$ -smooth. Indeed, in this case the Gaussian curvature of



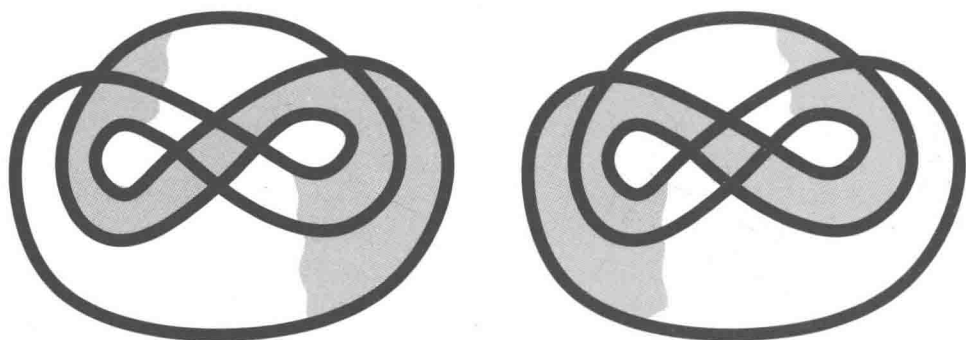


Figure 0.2. The immersions  $f$  and  $g$ .

the metric on  $S^2$  should be  $\geq 4$  at least somewhere. However, surprisingly, the answer is “yes” in the case of  $C^1$ -immersions (when the curvature is not defined but the curve length is), see [Na54, Ku55] and Chapter 21 below.

**C. Mappings with a prescribed Jacobian.** Let  $\Omega$  be an  $n$ -form on a closed oriented stably parallelizable  $n$ -dimensional manifold  $M$  such that  $\int_M \Omega = 0$ , and let

$$\eta = dx_1 \wedge \cdots \wedge dx_n$$

be the standard volume form on  $\mathbb{R}^n$ . Then there exists a map  $f : M \rightarrow \mathbb{R}^n$  such that  $f^*\eta = \Omega$ . See [GE73]. ►

All the above statements are examples of the *homotopy principle*, or the *h-principle*. Despite the fact that all these problems are asking for the solution of certain differential equations or inequalities, they can be reduced to problems of a pure homotopy-theoretic nature which then can be dealt with using the methods of Algebraic topology. For instance, the regular homotopy classification of immersions  $S^2 \rightarrow \mathbb{R}^3$  can be reduced to the computation of the homotopy group  $\pi_2(\mathbb{R}P^3)$ , which is trivial.

We are teaching in this book how to deal with these problems. In particular, two general methods which we describe here will be sufficient to handle all the above examples, except **A3** and **C**. In our sequel book, “The  $h$ -Principle and Singularities”, we will discuss other methods which prove, in particular, the two remaining results.

Another, maybe even more important, goal of this book is to teach the reader *how to recognize* the problems which may satisfy the  $h$ -principle. Of course, in the most interesting cases this is a very difficult question. As we will see below there are plenty of open problems where one neither can establish the  $h$ -principle, nor find a single instance of *rigidity*. Nevertheless