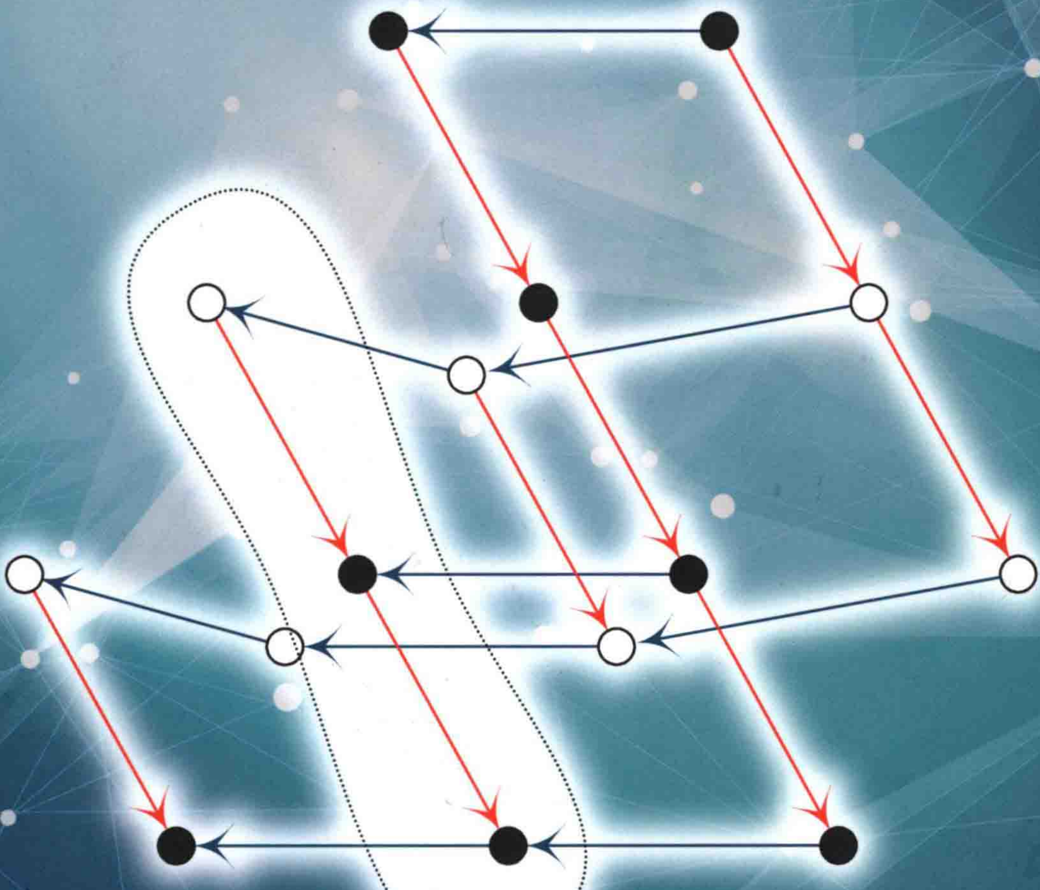
CRYSTAL BASES

Representations and Combinatorics



Daniel Bump

Anne Schilling

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Representations and Combinatorics

This unique book provides the first introduction to crystal base theory from the combinatorial point of view. Crystal base theory was developed by Kashiwara and Lusztig from the perspective of quantum groups. Its power comes from the fact that it addresses many questions in representation theory and mathematical physics by combinatorial means. This book approaches the subject directly from combinatorics, building crystals through local axioms (based on ideas by Stembridge) and virtual crystals. It also emphasizes parallels between the representation theory of the symmetric and general linear groups and phenomena in combinatorics. The combinatorial approach is linked to representation theory through the analysis of Demazure crystals. The relationship of crystals to tropical geometry is also explained.

World Scientific
www.worldscientific.com
9876 hc

ISBN 978-981-4733-43-4
9 789814 733434

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Bump
Schilling

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Representations and Combinatorics

Daniel Bump
Stanford University, USA

Anne Schilling
University of California, Davis, USA

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NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI • TOKYO

Published by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data
Names: Bump, Daniel, 1952– | Schilling, Anne (Mathematician)
Title: Crystal bases : representations and combinatorics / by Daniel Bump (Stanford), Anne Schilling (UC Davis).
Description: New Jersey : World Scientific, 2016. | Includes bibliographical references and index.
Identifiers: LCCN 2016047620| ISBN 9789814733434 (hardcover : alk. paper) | ISBN 9789814733441 (pbk. : alk. paper)
Subjects: LCSH: Lie algebras. | Quantum groups. | Combinatorial analysis.
Classification: LCC QA252.3 .B86 2016 | DDC 512/.482--dc23
LC record available at https://lccn.loc.gov/2016047620

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

Printed in Singapore

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Preface

Crystal bases are purely combinatorial objects that are analogous to representations of Lie groups or Lie algebras. They appeared in the works of Kashiwara, Lusztig and Littelmann on quantum groups and the geometry of flag varieties. In retrospect, topics from the combinatorical theory of tableaux such as the famous Robinson–Schensted–Knuth algorithm and the plactic monoid of Lascoux and Schützenberger fit into the crystal base theory. Crystal bases come up in many unexpected places, from mathematical physics to number theory.

This book originated from a plan to approach crystal base theory from a purely combinatorial point of view. It is aimed at graduate students and researchers who wish to delve into this subject.

It seems that every exposition of crystal base theory needs some powerful method behind the proofs. In existing expositions on crystal bases such as [Hong and Kang (2002)], [Kashiwara (2002)] and Littelmann [Littelmann (1997)] this has come from either quantum groups or Littelmann paths. We have taken a different path, relying on ideas of Stembridge and Kashiwara for our foundations. Thus we were able to prove everything combinatorially. In our approach, the link between crystals and representation theory is made through Demazure crystals.

It is assumed that the reader is familiar with root systems, their classifications, Coxeter groups, and Cartan types, part of which are reviewed in Chapter 2. A bit of algebraic geometry knowledge will also be helpful in Chapter 15. In order to help the reader appreciate certain analogies between crystal bases and representation theory, we have included two appendices on standard topics in the representation theory of Lie groups.

Preliminary versions of this book were used for a short lecture series at NCSU in October 2015, for a quarter special topics class at Stanford and a reading class at UC Davis in the Winter Quarter of 2016. We are grateful for lots of help from various people, as explained in the Acknowledgments.

Daniel Bump
Anne Schilling

California, August 2016

Acknowledgments

We would like to thank Graham Hawkes, Patricia Hersh, Henry Kvinge, Peter Littelmann, Neal Livesay, Molly Lynch, Shotaro Makisumi, Eric Marberg, Kirill Paramonov, Arun Ram, Ben Salisbury, Travis Scrimshaw, Peter Tingley, Kurt Trampel, and Andrew Waldron for discussion and/or helpful comments on an earlier version of this book. Special thanks to Peter Tingley for help with Chapter 15, in particular with the exercises. And we would like to thank Ms. Kwong Lai Fun and World Scientific for their help with the manuscript.

The second author would like to thank Kailash C. Misra, Daniel Nakano, Brian Parshall, and Weiqiang Wang for the invitation as a principal speaker to the 8th Southeastern Lie Theory Workshop on "Algebraic and Combinatorial Representation Theory" (October 9-11, 2015), where she gave a series of lectures on an early draft of this book.

This work was in part supported by NSF grants OCI-1147463, DMS–1001079 and DMS–1601026 (Daniel Bump), and OCI–1147247 and DMS–1500050 (Anne Schilling).

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Chapter 1

Introduction

Crystal bases or *Kashiwara crystals* are combinatorial structures that mirror representations of Lie groups. Historically, crystal bases were developed independently around 1990 from two independent sources.

On the one hand, [Kashiwara (1990, 1991, 1994)] showed that modules of quantum groups have "crystal bases" with remarkable combinatorial properties. Independently, [Lusztig (1990a,b)] introduced canonical bases from a more geometric perspective. Quantum groups are Hopf algebras that are "noncommutative" analogs of Lie groups. A particular class of quantum groups, *quantized enveloping algebras*, are deformations (in the category of Hopf algebras) of the universal enveloping algebras of Lie groups. They were described independently by [Drinfel′d (1985)] and [Jimbo (1985)] to explain developments in mathematical physics. Every representation of the Lie group gives rise to a representation of the corresponding quantized enveloping algebra, and Kashiwara showed that these modules have *crystal bases* whose properties he axiomatized and proved, using deep methods from quantum groups.

On the other hand, crystals also came about through the analysis of [Littelmann (1994, 1995b)] of *standard monomial theory* ([Lakshmibai, Musili and Seshadri (1979); Lakshmibai and Seshadri (1991)]). Borel and Weil and later [Bott (1957)] showed that representations of Lie groups can be realized as sections of line bundles on flag varieties. [Demazure (1974, 1976)] had found additional structure in these modules. Inspired by work of [Hodge (1943)] on the cohomology of Grassmannians, Seshadri and Lakshmibai found convenient bases of these modules of sections that are indexed by tableaux. Peter Littelmann, in the early 1990's, reinterpreted these bases as paths through a vector space containing the weight lattice and showed that they may be organized into crystals like those found by Kashiwara in the theory of quantum groups. [Kashiwara (1996); Joseph (1995)] then proved that the crystals arising from quantum groups are the same as the crystals arising from the *Littelmann paths*.

In retrospect, some older work in the combinatorics of tableaux can be understood in terms of crystals. [Littlewood (1940)] showed that a *Schur polynomial*, which is the character of an irreducible representation of $\mathrm{GL}(n)$, had a combinatorial definition as a sum over tableaux. In fact, both the irreducible representations of the symmetric group and the general linear group were known to have bases indexed by tableaux, and the *Robinson–Schensted–Knuth* (RSK) algorithm [Knuth (1970, 1998)] gave bijections that are combinatorial analogs of certain isomorphisms between such modules of Lie groups and the symmetric groups. Later, [Lascoux and Schützenberger (1981)] gave a multiplicative structure on the set of tableaux, called the *plactic monoid*, that is closely related to RSK. All of these topics fit into the theory of crystal bases and the connections will be discussed in Chapters 7 and 8.

Crystals appear in many other contexts from mathematical physics and combinatorics to number theory. We will not attempt to survey all of these here.

In this book, we will limit ourselves to crystals associated to finite-dimensional Lie algebras, omitting the important topic of crystals of representations of infinite-dimensional Lie algebras. Within this limited scope, we have tried to prove the essential facts using combinatorial methods. The facts one wants to prove are as follows.

Given a reductive complex Lie group G, there is an associated *weight lattice* Λ with a cone of *dominant weights*. Given a dominant weight λ, there is a unique irreducible representation of *highest weight* λ. There are two operations on these that we are particularly concerned with: *tensor product* of representations and *branching*, or restriction, to Levi subgroups.

In the theory of crystal bases, one starts with the same weight lattice and cone of dominant weights. Instead of a representation, one would like to associate a special crystal to each dominant weight. If the representation is irreducible, the crystal should be connected. There may be many connected crystals with a given highest weight, but it turns out that there is one particular one that we call *normal*. We think of this as the "crystal of the representation." More generally, a crystal that is the disjoint union of such crystals, is to be considered normal.

The operations of tensor product and Levi branching from representation theory also make sense for crystals. The usefulness of the class of normal crystals is that the decomposition of a crystal into irreducibles with respect to these operations is again normal. Moreover, the decomposition of a representation obtained by tensoring representations or branching a represention to a Levi subgroup gives the same multiplicities as the decomposition of the tensor product or Levi branching of the corresponding normal crystals into irreducibles.

There are several ways of defining normal crystals. [Kashiwara (1990, 1991, 1994)] and [Littelmann (1994, 1995b)] gave two different definitions, which then were shown to be equivalent. We give yet another definition of normal crystals, based on two key ideas: *Stembridge crystals* ([Stembridge (2003)]) and *virtual crystals* ([Kashiwara (1996); Baker (2000)]). For the simply-laced Cartan types, [Stembridge (2003)] showed how to characterize the normal crystals axiomatically. This is subject of Chapter 4. This approach does not work as well for the non-simply-laced types, but for these, there is a way of embedding certain crystals into crystals of