



海外优秀数学类教材系列丛书

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Advanced Engineering Mathematics

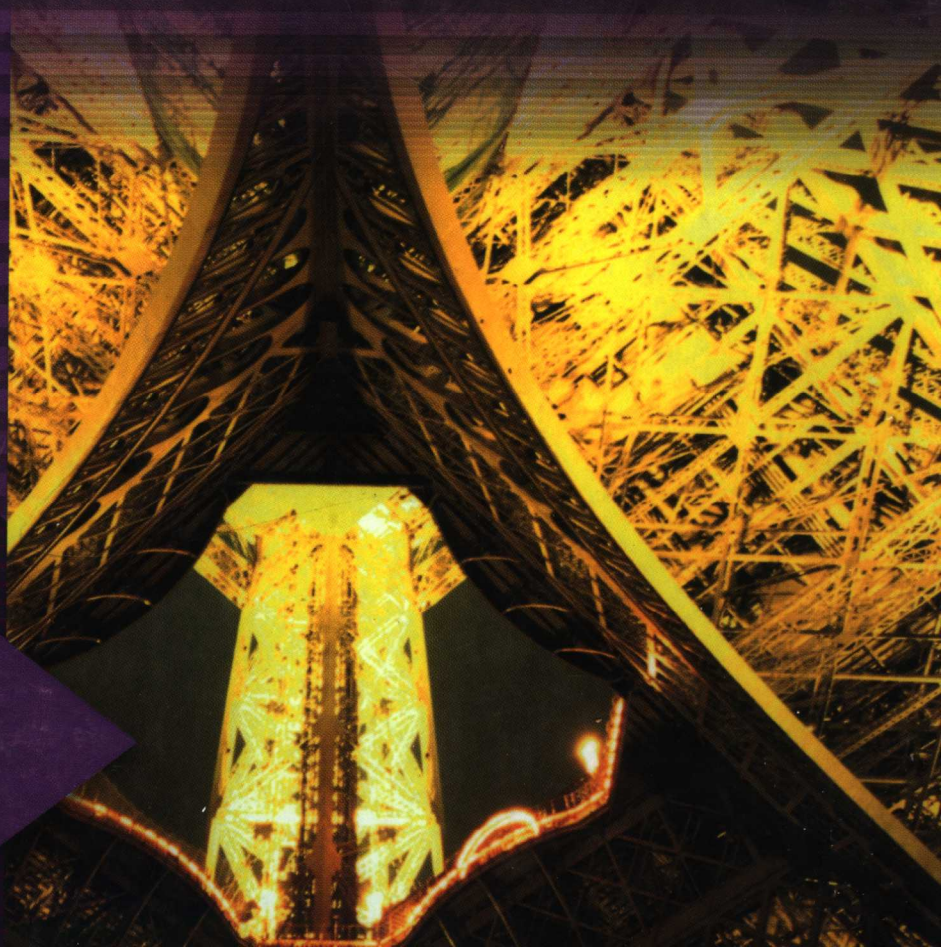
(Fifth Edition)

高等工程数学 (第5版) (上册)

□ Peter V. O'Neil



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Higher Education Press



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Advanced Engineering Mathematics

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高等工程数学 (第5版) (上册)

Peter V. O'Neil

University of Alabama at Birmingham

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出版者的话

在我国已经加入 WTO、经济全球化的今天,为适应当前我国高校各类创新人才培养的需要,大力推进教育部倡导的双语教学,配合教育部实施的“高等学校教学质量与教学改革工程”和“精品课程”建设的需要,高等教育出版社有计划、大规模地开展了海外优秀数学类系列教材的引进工作。

高等教育出版社和 Pearson Education, John Wiley & Sons, McGraw-Hill, Thomson Learning 等国外出版公司进行了广泛接触,经国外出版公司的推荐并在国内专家的协助下,提交引进版权总数 100 余种。收到样书后,我们聘请了国内高校一线教师、专家、学者参与这些原版教材的评介工作,并参考国内相关专业的课程设置和教学实际情况,从中遴选出了这套优秀教材组织出版。

这批教材普遍具有以下特点:(1)基本上是近 3 年出版的,在国际上被广泛使用,在同类教材中具有相当的权威性;(2)高版次,历经多年教学实践检验,内容翔实准确、反映时代要求;(3)各种教学资源配套整齐,为师生提供了极大的便利;(4)插图精美、丰富,图文并茂,与正文相辅相成;(5)语言简练、流畅、可读性强,比较适合非英语国家的学生阅读。

本系列丛书中,有 Finney、Weir 等编的《托马斯微积分》(第 10 版, Pearson),其特色可用“呈传统特色、富革新精神”概括,本书自 20 世纪 50 年代第 1 版以来,平均每四五年就有一个新版面世,长在 50 余年始终盛行于西方教坛,作者既有相当高的学术水平,又热爱教学,长期工作在教学第一线,其中,年近 90 的 G.B.Thomas 教授长年在 MIT 工作,具有丰富的教学经验;Finney 教授也在 MIT 工作达 10 年;Weir 是美国数学建模竞赛委员会主任。Stewart 编的立体化教材《微积分》(第 5 版, Thomson Learning)配备了丰富的教学资源,是国际上最畅销的微积分原版教材,2003 年全球销量约 40 余万册,在美国,占据了约 50%~60%的微积分教材市场,其用户包括耶鲁等名牌院校及众多一般院校 600 余所。本系列丛书还包括 Anton 编的经典教材《线性代数及其应用》(第 8 版, Wiley); Jay L. Devore 编的优秀教材《概率论与数理统计》(第 5 版, Thomson Learning)等。在努力降低引进教材售价方面,高等教育出版社做了大量和细致的工作,这套引进的教材体现了一定的权威性、系统性、先进性和经济性等特点。

通过影印、翻译、编译这批优秀教材,我们一方面要不断地分析、学习、消化吸收国外优秀教材的长处,吸取国外出版公司的制作经验,提升我们自编教材的立体化配套标准,使我国高校教材建设水平上一个新的台阶;与此同时,我们还将尝试组织海外作者和国内作者合编外文版基础课数学教材,并约请国内专家改编部分国外优秀教材,以适应我国实际教学环境。

这套教材出版后,我们将结合各高校的双语教学计划,开展大规模的宣传、培训工作,及时地将本套丛书推荐给高校使用。在使用过程中,我们衷心希望广大高校教师和同学提出宝贵的意见和建议,如有好的教材值得引进,请与高等教育出版社高等理科分社联系。

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Preface

This Fifth Edition of *Advanced Engineering Mathematics* has two primary objectives.

The first is to make available much of the post-calculus mathematics needed and used by today's scientists, engineers, and applied mathematicians, in a setting that is helpful to both students and faculty. Throughout, it is recognized that mathematics provides powerful ways of modeling physical processes, but that these can lead to false conclusions if misunderstood or misapplied. This warrants careful attention to details in making correct statements of theorems and methods and in analyzing results.

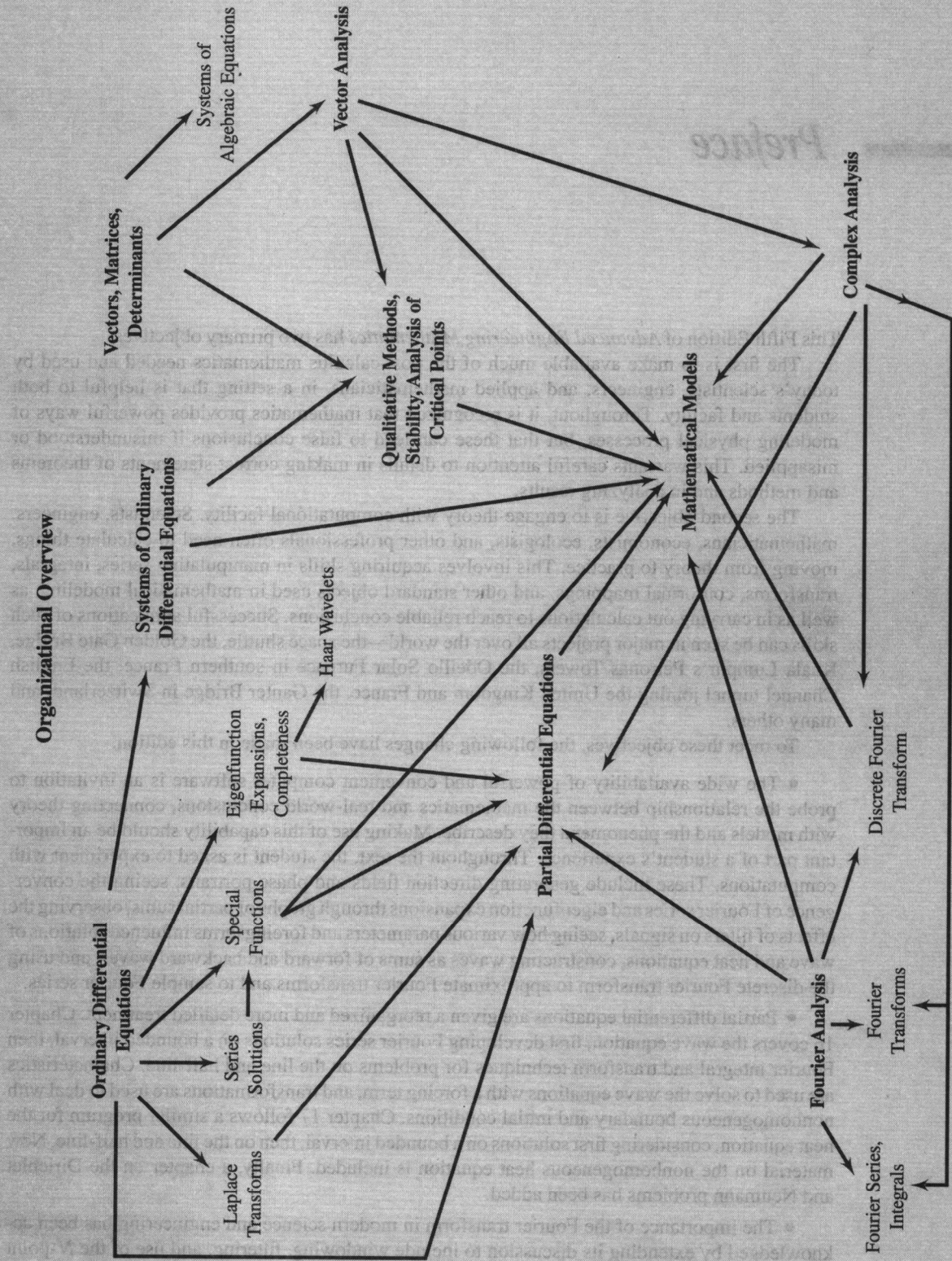
The second objective is to engage theory with computational facility. Scientists, engineers, mathematicians, economists, ecologists, and other professionals often need to calculate things, moving from theory to practice. This involves acquiring skills in manipulating series, integrals, transforms, conformal mappings, and other standard objects used in mathematical modeling, as well as in carrying out calculations to reach reliable conclusions. Successful applications of such skills can be seen in major projects all over the world—the space shuttle, the Golden Gate Bridge, Kuala Lumpur's Petronas Towers, the Odeillo Solar Furnace in southern France, the English Channel tunnel joining the United Kingdom and France, the Ganter Bridge in Switzerland, and many others.

To meet these objectives, the following changes have been made in this edition.

- The wide availability of powerful and convenient computer software is an invitation to probe the relationship between the mathematics and real-world conclusions, connecting theory with models and the phenomena they describe. Making use of this capability should be an important part of a student's experience. Throughout the text, the student is asked to experiment with computations. These include generating direction fields and phase portraits, seeing the convergence of Fourier series and eigenfunction expansions through graphs of partial sums, observing the effects of filters on signals, seeing how various parameters and forcing terms influence solutions of wave and heat equations, constructing waves as sums of forward and backward waves, and using the discrete Fourier transform to approximate Fourier transforms and to sample Fourier series.

- Partial differential equations are given a reorganized and more detailed treatment. Chapter 16 covers the wave equation, first developing Fourier series solutions on a bounded interval, then Fourier integral and transform techniques for problems on the line and half-line. Characteristics are used to solve the wave equations with a forcing term, and transformations are used to deal with nonhomogeneous boundary and initial conditions. Chapter 17 follows a similar program for the heat equation, considering first solutions on a bounded interval, then on the line and half-line. New material on the nonhomogeneous heat equation is included. Finally, a chapter on the Dirichlet and Neumann problems has been added.

- The importance of the Fourier transform in modern science and engineering has been acknowledged by extending its discussion to include windowing, filtering, and use of the N -point discrete Fourier transform to sample Fourier series and approximate Fourier transforms.



- Special functions are given a more thorough, unified treatment. Legendre polynomials, Bessel functions, and then other orthogonal polynomials are placed in the context of Sturm–Liouville theory and general eigenfunction expansions, with specific examples. These are followed by mean square convergence, the significance of completeness of the eigenfunctions, and the relationship between completeness and Parseval’s theorem. Haar wavelets are discussed in the context of completeness and orthogonal expansions.

- The treatment of systems of linear, ordinary differential equations has additional material on the case of repeated eigenvalues of the coefficient matrix.

- The discussion of the qualitative behavior of nonlinear systems has more details on phase portraits and the classification of critical points and stability, including Lyapunov’s criteria.

- The sections on determinants have been reorganized to provide greater clarity in understanding properties of determinants.

- The section containing answers to odd-numbered problems has been expanded, including details for some of the more difficult problems, as well as more illustrations.

- Behind the mathematics we usually find interesting people and events. Some of their stories are told in Chapters 26 and 27, but historical perspectives are also included throughout the text. For example, Section 14.9.1, on the fast Fourier transform, begins with a review of the personal interactions that led to the publication of the famous Cooley–Tukey paper, the first detailed description of the algorithm.

The chart opposite offers a complete organizational overview.

Acknowledgments

The production of a book of this size and scope requires much more than an author. Among those to whom I owe a debt of appreciation are Bill Stenquist, Pat Call, and Mary Vezilich of Brooks/Cole, Martha Emry of Martha Emry Production Services, designer Terri Wright, and the professionals at Techsetters, Inc., and Laurel Technical Services.

Dr. Thomas O’Neil of the California Polytechnic State University contributed material to previous editions, and much of this is continued in this edition. Dr. Fred Martens of the University of Alabama has helped with error checking in both text and problems and is the author of the solutions manual accompanying this edition. Finally, I want to acknowledge Rich Jones, who had the vision for the first edition of this text many years ago.

I would also like to acknowledge my debt to the reviewers, whose suggestions for improvements and clarifications are much appreciated:

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$$\mathcal{L}[f](s) = \int_0^{\infty} e^{-st} t^{-1/2} dt = 2 \int_0^{\infty} e^{-sx^2} dx \text{ (set } x = t^{1/2}) = \frac{2}{\sqrt{s}} \int_0^{\infty} e^{-z^2} dz \text{ (set } z = x\sqrt{s})$$

PART 1

$$= 2 \int_0^{\infty} e^{-sx^2} dx$$

$$= \sqrt{\frac{\pi}{s}}$$

CHAPTER 1

First-Order Differential Equations

CHAPTER 2

Second-Order Differential Equations

CHAPTER 3

The Laplace Transform

CHAPTER 4

Series Solutions

$$\mathcal{L}[f](s) = \int_0^{\infty} e^{-st} t^{-1/2} dt$$

$$= 2 \int_0^{\infty} e^{-sx^2} dx \text{ (set } x = t^{1/2})$$

$$= \frac{2}{\sqrt{s}} \int_0^{\infty} e^{-z^2} dz \text{ (set } z = x\sqrt{s})$$

$$\mathcal{L}[f](s) = \int_0^{\infty} e^{-st} t^{-1/2} dt$$

$$= 2 \int_0^{\infty} e^{-sx^2} dx$$

$$= \frac{2}{\sqrt{s}} \int_0^{\infty} e^{-z^2} dz$$

$$= \sqrt{\frac{\pi}{s}}$$

Ordinary Differential Equations

A *differential equation* is an equation that contains one or more derivatives. For example,

$$y''(x) + y(x) = 4 \sin(3x)$$

and

$$\frac{d^4 w}{dt^4} - (w(t))^2 = e^{-t}$$

are differential equations. These are *ordinary* differential equations because they involve only total derivatives, rather than partial derivatives.

Differential equations are interesting and important because they express relationships involving rates of change. Such relationships form the basis for developing ideas and studying phenomena in the sciences, engineering, economics, and, increasingly, in other areas, such as the business world and the stock market. We will see examples of applications as we learn more about differential equations.

The *order* of a differential equation is the order of its highest derivative. The first example given above is of second order, while the second is of fourth order. The equation

$$xy' - y^2 = e^x$$

is of first order.

A *solution* of a differential equation is any function that satisfies it. A solution may be defined on the entire real line, or on only part of it, often an interval. For example,

$$y = \sin(2x)$$

is a solution of

$$y'' + 4y = 0,$$

because, by direct differentiation,

$$y'' + 4y = -4 \sin(2x) + 4 \sin(2x) = 0.$$

This solution is defined for all x (that is, on the whole real line).

By contrast,

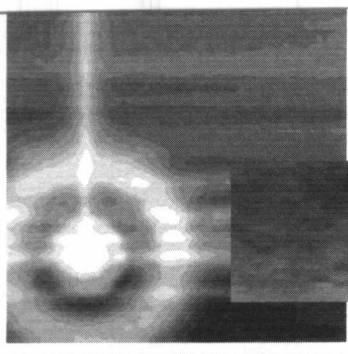
$$y = x \ln(x) - x$$

is a solution of

$$y' = \frac{y}{x} + 1,$$

but this solution is defined only for $x > 0$. Indeed, the coefficient $1/x$ of y in this equation means that $x = 0$ is disallowed from the start.

We now begin a systematic development of ordinary differential equations, starting with the first-order case.



CHAPTER 1

First-Order Differential Equations

PRELIMINARY CONCEPTS SEPARABLE EQUATION
HOMOGENEOUS, BERNOULLI, AND RICCATI EQUATIONS APPLICATIONS TO MECHANICS, ELECTRIC CIRCUITS, AND ORTHOGONAL TRAJECTORIES EX

1.1 Preliminary Concepts

Before developing techniques for solving various kinds of differential equations, we will develop some terminology and geometric insight.

1.1.1 General and Particular Solutions

A first-order differential equation is any equation involving a first derivative, but no higher derivative. In its most general form, it has the appearance

$$F(x, y, y') = 0, \quad (1.1)$$

in which $y(x)$ is the function of interest and x is the independent variable. Examples are

$$y' - y^2 - e^y = 0,$$

$$y' - 2 = 0,$$

and

$$y' - \cos(x) = 0.$$

Note that y' must be present for an equation to qualify as a first-order differential equation, but x and/or y need not occur explicitly.

A solution of equation (1.1) on an interval I is a function ϕ that satisfies the equation for all x in I . That is,

$$F(x, \phi(x), \phi'(x)) = 0 \quad \text{for all } x \text{ in } I.$$

For example,

$$\phi(x) = 2 + ke^{-x}$$

is a solution of

$$y' + y = 2$$

for all real x and for any number k . Here I can be chosen as the entire real line. And

$$\varphi(x) = x \ln(x) + cx$$

is a solution of

$$y' = \frac{y}{x} + 1$$

for all $x > 0$ and for any number c .

In both of these examples, the solution contained an arbitrary constant. This is a symbol independent of x and y that can be assigned any numerical value. Such a solution is called the *general solution* of the differential equation. Thus,

$$\varphi(x) = 2 + ke^{-x}$$

is the general solution of $y' + y = 2$.

Each choice of the constant in the general solution yields a *particular solution*. For example,

$$f(x) = 2 + e^{-x}, \quad g(x) = 2 - e^{-x},$$

and

$$h(x) = 2 - \sqrt{53}e^{-x}$$

are all particular solutions of $y' + y = 2$, obtained by choosing, respectively, $k = 1, -1$, and $-\sqrt{53}$ in the general solution.

1.1.2 Implicitly Defined Solutions

Sometimes we can write a solution explicitly giving y as a function of x . For example,

$$y = ke^{-x}$$

is the general solution of

$$y' = -y,$$

as can be verified by substitution. This general solution is explicit, with y isolated on one side of an equation and a function of x on the other.

By contrast, consider

$$y' = -\frac{2xy^3 + 2}{3x^2y^2 + 8e^{4y}}.$$

We claim that the general solution is the function $y(x)$ implicitly defined by the equation

$$x^2y^3 + 2x + 2e^{4y} = k, \tag{1.2}$$

in which k can be any number. To verify this, implicitly differentiate equation (1.2) with respect to x , remembering that y is a function of x . We obtain

$$2xy^3 + 3x^2y^2y' + 2 + 8e^{4y}y' = 0,$$

and solving for y' yields the differential equation.

In this example we are unable to solve equation (1.2) explicitly for y as a function of x , isolating y on one side. Equation (1.2), implicitly defining the general solution, was obtained by a technique we will develop shortly, but this technique cannot guarantee an explicit solution.