INTRODUCTION TO FINITE ELEMENT, BOUNDARY ELEMENT, AND MESHLESS METHODS

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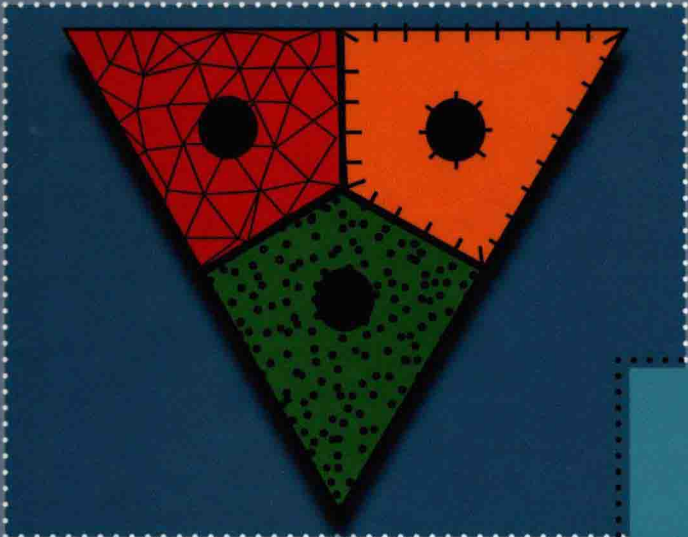

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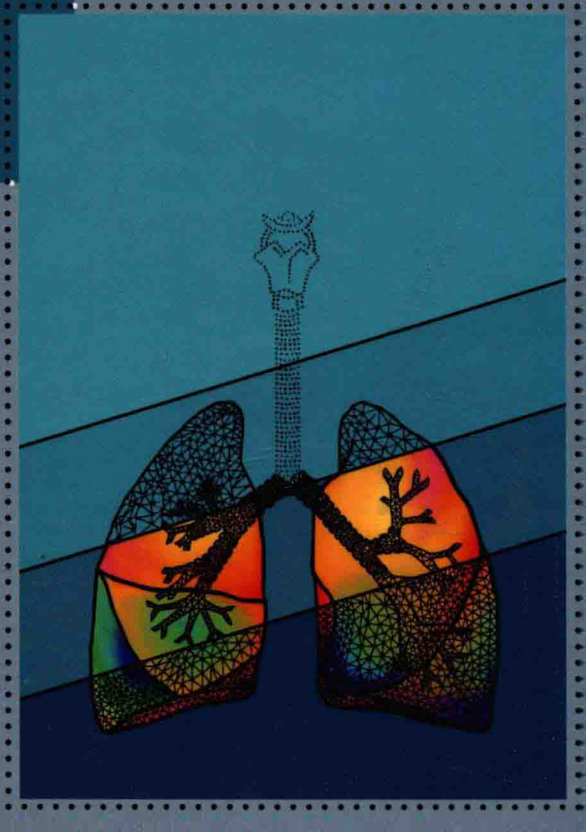

With Applications to Heat Transfer and Fluid Flow

AN INTRODUCTION TO FINITE ELEMENT, BOUNDARY ELEMENT, AND MESHLESS METHODS

With Applications to Heat Transfer and Fluid Flow

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DEDICATION

To the students and masters of these elegant numerical methods, as well as future numerical methods yet to come.

PREFACE

This book stems from our experiences in teaching numerical methods to both engineering students and experienced, practicing engineers in industry. The emphasis in this book deals with finite element, boundary element, and meshless methods. Much of the material comes from courses we have conducted over many years at our institutions, including AIAA home study and ASME short courses presented over several decades, as well as from the suggestions and recommendations of our colleagues and students. There are numerous books on applied numerical methods, many of them being finite element and boundary element textbooks available in the literature today. However, there are very few books dealing with meshless methods, especially those showing how nearly all of these numerical schemes originate from the fundamental principles of the method of weighted residuals. We find that when students once master the concepts of the finite element method (and meshing), it's not long before they begin to look at more advanced numerical techniques and applications, especially the boundary element and meshless methods (since a mesh is not required). Our intent in this book is to provide a simple explanation of these three powerful numerical schemes, and to show how they all fall under the umbrella of the more universal method of weighted residuals approach.

The book is divided into three sections, beginning with the finite element method, then progressing through the boundary element method, and finally ending with the meshless method. Each section serves as a stand-alone description, but it is apparent to see how each conveniently leads to the other techniques. We recommend that the reader begin with the finite element method, as this serves as the primary basis for defining the method of weighted residuals.

We begin by introducing the basic fundamentals of the finite element method using simple examples. Particular attention is given to the development of the discrete set of algebraic equations, beginning with simple one-dimensional problems that can be solved by inspection, and continuing to two- and three-dimensional elements. Once these principles are grasped, we then introduce the concept of boundary elements, and the relative ease with which one reduces the dimensionality of a problem (a great relief when solving large problems, or problems with infinite domain boundaries). The boundary element technique is a natural extension of the finite element method, and becomes greatly appreciated by users. While the method has some limitations regarding the wide range of applications afforded by the finite element technique, it is still a very popular and useful method. It is finding use in crack growth and related applications dealing with structural mechanics, and couples nicely with finite element meshes.

The more recent introduction of meshless methods is rapidly becoming a method now being used by practitioners of both finite element and boundary element methods. The method is simple to grasp, and simple to implement. The power of the method is becoming more appreciated with time. The meshless method has been shown to yield solutions with accuracies comparable to finite element methods employing an extensive number of elements, yet requiring no mesh (or connectivity of nodes). While there is much left to discover with regards to some of the formulation and parameters used in the development of the meshless method, it is a method with much promise and wide spread applications. We have used it for structural analysis, fluid flow, heat transfer, and various biomedical applications.

We provide computer files in both MathCad and MATLAB that are used to illustrate the setup and subsequent solutions of these example problems. These computer codes are not elegant nor optimized for efficiency, but do provide the reader with the logic and steps necessary to obtain solutions. The code listings are available from the www.fbm.centecorp.com website, along with example data files.

There are many commercially available finite element codes available in the market, and a few that are free via the web. We tend to use COMSOL because of its ease of use, and it multiphysics capabilities. COMSOL is a very versatile finite element code that handles a wide variety of applications, including fluid flow, heat transfer, solid mechanics, and electrodynamics. This package runs on PCs.

Because many finite element and boundary element books are written for the structurally oriented engineer, those nonstructural engineers and students more interested in the fluid-thermal fields must sift through undesired concepts and applications before finding a relevant problem area. We have found that students quickly grasp the basic concepts of heat transfer and can easily follow the principles of heat flow and one degree of freedom (temperature). A simple generic approach is utilized in this book that is focused on the transport and diffusion of heat (scalar transport); we then illustrate how one can extend these basic approaches to wider applications, with emphasis on the nonlinear equations for fluid motion.

We wish to thank our colleagues and former students who have greatly contributed to the material presented in this book. We began some years ago by offering several free short courses stemming from the information within this book to our colleagues in the ASME Heat Transfer Division. We gaged their reactions and interests, and have incorporated their suggestions in arranging the presentation of information and material. We especially wish to thank Erik Pepper and Mrs. Julie Longo for their efforts in editing the manuscript and graphical images in this book, and to our ASME Press Editor, Mary Grace Stefanchik, for her helpful comments and editorial assistance; we also wish to thank our former students and colleagues for their patience in reading and suggestions for revising the manuscript.

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September 9, 2014

Overview

THE METHOD OF WEIGHTED RESIDUALS (MWR)

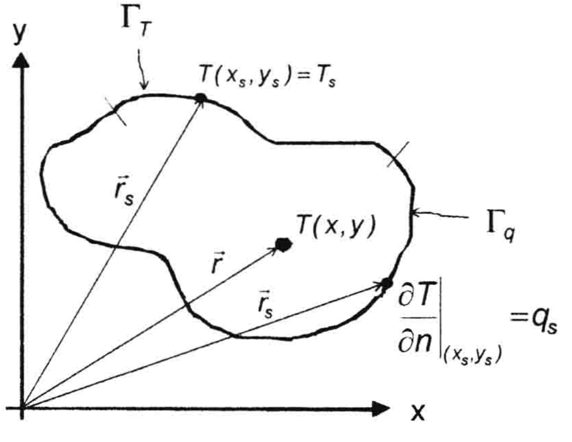
This book is focused on three numerical methods utilized for analysis of field problems in heat transfer and fluid flow: the Finite Element Method (FEM), the Boundary Element Method (BEM), and the Meshless Method (MM). The three numerical methods discussed in this book, and for that matter, most of the other commonly utilized numerical methods, including Finite Difference Method (FDM) [1] and Finite Volume Method (FVM) [2,3], can be formulated in the single over-arching framework of the Method of Weighted Residuals (MWR) [4].

In order to develop the MWR formulation, let us consider a typical steady-state heat transfer problem where the temperature, $T(x,y)$, is governed by the heat conduction equation and subjected to either first kind (prescribed temperature) or second kind (prescribed temperature gradient) boundary conditions,

G.E.: $\nabla^2 T(x,y) + u_G = 0 \qquad \vec{r} \in \Omega$

B.C.'s: $T(x_s, y_s) = T_s \qquad \vec{r}_s \in \Gamma_T$

$\left.\dfrac{\partial T}{\partial n}\right|_{(x_s,y_s)} = q_s \qquad \vec{r}_s \in \Gamma_q$



where $\vec{r}_s$ is the position vector to a point (x_s,y_s) on the boundary Γ binding a domain Ω. As a note, we choose this problem as an illustrative example, and the procedure we now outline can apply to any other governing scalar or vector linear or non-linear equation subject to any other type of boundary condition not listed above.

The basic premise of MWR is to approximate the temperature by a set of trial functions, $\phi_j(x, y)$, as

$$\tilde{T}(x,y) = \sum_{j=1}^{N} \alpha_j \phi_j(x,y) \tag{1}$$

We are free to choose to have localized or global support, with the only obvious requirement that the trial functions must be linearly independent. The expansion coefficients, α_j, may have physical meaning, such as representing nodal temperatures in FDM and FVM, or may be arbitrary.

Introducing Eq. (1) into the governing equation leads to a domain residual, $R_\Omega(x, y)$,

$$R_\Omega(x,y) = \nabla^2 \tilde{T}(x,y) + u_G \quad (x,y) \in \Omega \tag{2}$$

Introducing Eq. (1) into the boundary conditions leads to boundary residuals. In particular, this leads to a residual, $R_{\Gamma_T}(x,y)$, on the Γ_T portion of the boundary where a first kind boundary condition is imposed

$$R_{\Gamma_T}(x,y) = \tilde{T}(x_s, y_s) - T_s \qquad (x_s, y_s) \in \Gamma_T \tag{3}$$

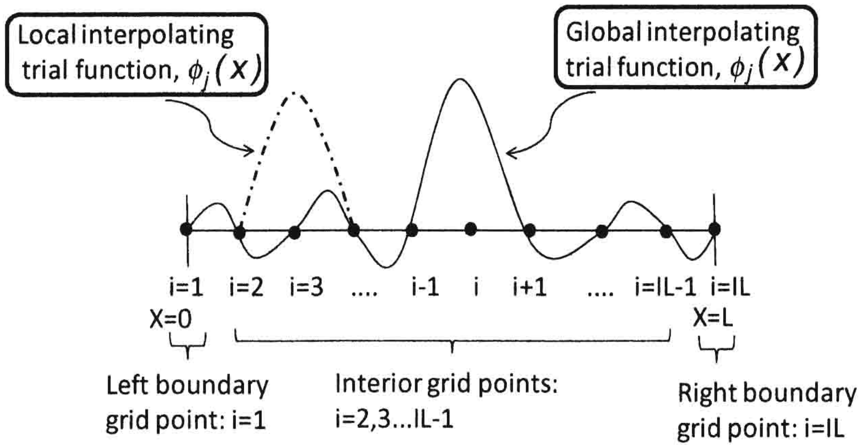


Figure 1. Illustration of 1-D local and global trial functions, $\phi_j(x)$.

and to a residual, $R_{\Gamma_q}(x,y)$, on the Γ_q portion of the boundary where a second kind boundary condition is imposed

$$R_{\Gamma_q}(x,y) = \frac{\partial \tilde{T}(x_s,y_s)}{\partial n} - q_s \qquad (x_s,y_s) \in \Gamma_q \tag{4}$$

Depending on our *choice of trial functions* any of these residuals may be zero, and that choice broadly differentiates numerical methods from a MWR perspective as being:

1. *Interior methods:* trial functions satisfy the boundary conditions, and this leads to a domain residual only.
2. *Boundary methods:* trial functions satisfy the governing equation, and this leads to boundary residuals only.
3. *Mixed methods:* trial functions satisfy neither the governing equation nor the boundary conditions, and this leads to both a domain and boundary residuals.

The FDM, FVM, and FEM are mixed methods with trial functions that have local support. The BEM is a boundary method, and the MM is a mixed method with trial functions that have, depending on the technique, either global or local support as referenced Fig. 1, with the latter the most widely used in practice.

The next task in MWR is to determine the unknown expansion coefficients by minimizing the residual. To this end, weighting functions are introduced: (a) a weighting function, $W_\Omega(x, y)$, for the domain residual $R_\Omega(x, y)$; (b) a weighting function, $w_{\Gamma_T}(x,y)$, for the boundary residual, $R_{\Gamma_T}(x,y)$, on portion Γ_T of the boundary; (c) a weighting function, $w_{\Gamma_q}(x,y)$, for the boundary residual, $R_{\Gamma_q}(x,y)$, on portion Γ_q of the boundary. A weighted residual statement is then formulated to solve for the expansion coefficients,

$$\oiint_{\Omega} R_\Omega(x,y) w_{\Omega,j}(x,y) d\Omega + \int_{\Gamma_T} R_{\Gamma_T}(x,y) w_{\Gamma_T,j}(x,y) d\Gamma + \int_{\Gamma_q} R_{\Gamma_q}(x,y) w_{\Gamma_q,j}(x,y) d\Gamma = 0 \quad j = 1,2...N \tag{5}$$

What further differentiates MWR techniques from each other is the *choice of the weighting functions* that leads to the following common minimization techniques:

1. **Collocation Method:** A set of collocation points, $\vec{r}_i$, is distributed on the domain and the boundary and the choice for the weighting function is the Dirac delta function, $\delta(\vec{r} - \vec{r}_i)$, acting at each one of these points,

$$w_j(\vec{r}) = \delta(\vec{r} - \vec{r}_j) \tag{6}$$

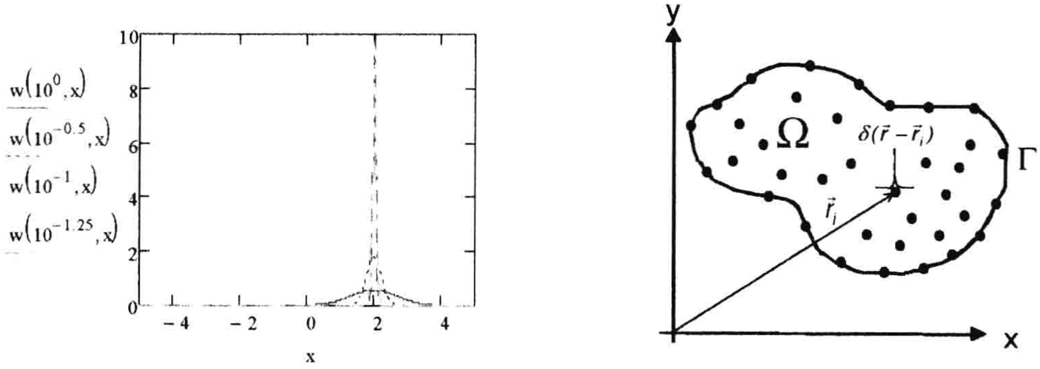


Figure 2. Plot of 1-D delta sequence acting at $x_i = 2$ and tending to Dirac delta function as $k \to 0$, and a Dirac delta function acts at a point (x_i, y_i) in a 2D domain.

The Dirac delta function is defined by its action on other functions, namely

$$\int_{-\infty}^{+\infty} \delta(x - x_i) f(x) dx = f(x_i) \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(x - x_i) dx = 1 \tag{7}$$

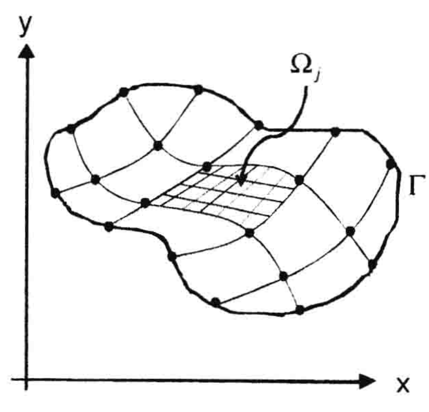
The Dirac delta function can be approximated numerically by any number of so-called delta sequences [6], for instance the following sequence obeys the property of a delta function in the given limit,

$$\delta(x - x_i) = \lim_{k \to 0} \left[\frac{e^{-\left(\frac{x - x_i}{k^2}\right)^2}}{\sqrt{\pi k}} \right] \tag{8}$$

as seen in Fig. 2. Multidimensional delta functions can be constructed as products of 1D delta functions. In Cartesian coordinates for instance: $\delta(x, y; x_i, y_i) = \delta(x - x_i)\delta(y - y_i)$. Collocation MWR is used to solve the governing equations in strong form and is the method employed to formulate the FDM and strong-form meshless methods. The FDM is a collocation MWR with local shape functions, typically taken as polynomials, the collocation points, $\vec{r}_i$, are called the mesh/grid points and are produced automatically by mesh generation procedures, the expansion coefficients, α_j, are the FDM nodal temperatures.

2. **Subdomain Method:** The domain Ω is subdivided into N-subdomains Ω_j, and the weighting function is chosen to be

$$\begin{aligned} w_j(\vec{r}) &= 1 \quad \text{if} \quad \vec{r} \in \Omega_j \\ &= 0 \quad \text{if} \quad \vec{r} \notin \Omega_j \end{aligned} \tag{9}$$



The FVM is a subdomain MWR with local shape functions, typically taken as polynomials, the subdomains are called finite volumes and are generated automatically by mesh generation techniques, and the expansion coefficients, α_j, are the FVM nodal temperatures.

3. **Galerkin Method:** The weighting function is chosen to be the expansion function itself, that is

$$w_j(\vec{r}) = \phi_j(\vec{r}) \tag{10}$$

The FEM is most often formulated using Galerkin MWR, using local shape functions, typically taken as polynomials, that are defined over a set of N-subdomains Ω_j called finite elements, and the expansion coefficients, α_j, are the FEM nodal temperatures.

4. **Least-Squares:** The weighting function is chosen to be the partial of the residual with respect to the expansion coefficients, α_j, that is

$$w_j(\vec{r}) = \frac{\partial R(\vec{r})}{\partial \alpha_j} \tag{11}$$

For example, supposing that we are considering a domain residual, we have

$$\oiint_\Omega R_\Omega(\vec{r}) \frac{\partial R_\Omega(\vec{r})}{\partial \alpha_j} d\Omega = \frac{1}{2} \frac{\partial}{\partial \alpha_j} \oiint_\Omega R_\Omega^2(\vec{r}) d\Omega \tag{12}$$

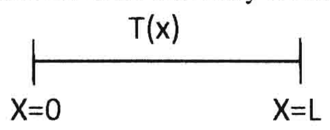
which is obviously a least-square minimization with respect to the expansion coefficients, α_j. There are some FEM formulations and meshless method formulations that utilize the concept of least-squares. There is another MWR formulation that minimizes using moments of the residual and the reader is referred to [4] for details on that method. The method of moments MWR finds applications as a numerical method in electromagnetics.

MWR EXAMPLE PROBLEM: FDM, FVM, FEM, BEM AND MM

Let us consider a simple 1D problem where the temperature is governed in a region $X \in [0, L]$ by the following non-homogeneous differential equation and first kind boundary conditions,

G.E.: $\dfrac{d^2T(x)}{dx^2} + T(x) + x = 0 \qquad x \in [0, L]$

B.C.'s: $T(0) = T_o$
$T(L) = T_L$



The exact solution to this problem is readily obtained as,

$$T(x) = T_o \cos(x) + \left(\frac{T_L + L - T_o \cos(L)}{\sin(L)} \right) \sin(x) - x \tag{13}$$

with the exact derivative of the temperature given by

$$q(x) = -T_o \sin(x) + \left(\frac{T_L + L - T_o \cos(L)}{\sin(L)} \right) \cos(x) - 1 \tag{14}$$

This temperature profile is illustrated in Fig. 3 for values of $T_o = 15$ and $T_L = 25$. We shall use this problem to illustrate the five numerical methods, FDM, FVM, FEM, BEM, and Localized Collocation Meshless Method (LCMM) formulated by the MWR principle corresponding to the particular method. The final result of the approximation process is an algebraic set of equations that are the discrete analog of the governing equation and boundary conditions that is solved by an appropriate numerical procedure.

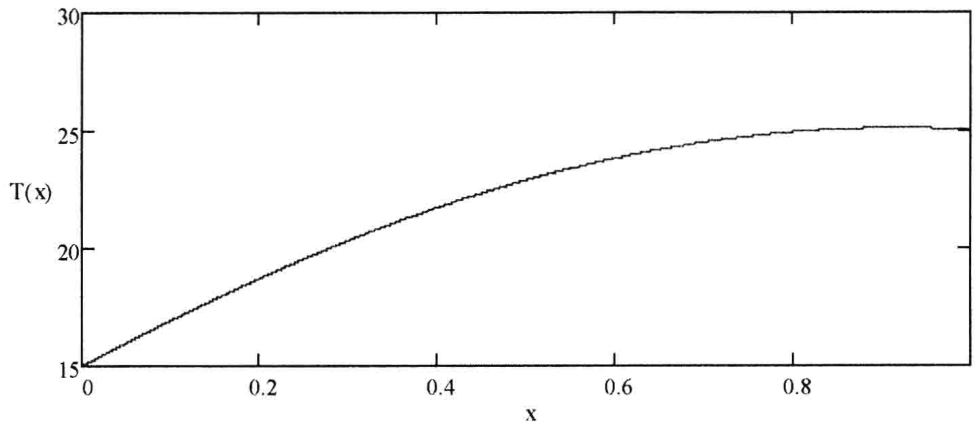


Figure 3. Temperature distribution for the MWR example problem with $T_o = 15$ and $T_L = 25$.

Finite Difference Method (FDM) – Collocation MWR with Local Polynomial Trial Functions

In the FDM [1], we lay out a set of i = 1,2...IL grid points to discretise the domain $x \in [0, L]$. This is usually accomplished by a grid generator. We identify the interior grid points, i = 2,3...IL – 1 and boundary grid points, i = 1 and i = IL. Here the grid spacing, $\Delta x = L/(IL-1)$, is uniform, although in general this is not the case as grid adaption is used to resolve regions of high gradients. The solution is sought at discrete locations, x_i, and denoted as $T(x_i) = T_i$, or the FDM nodal values of the temperature. Using collocation MWR, and placing the Dirac delta function at any interior node, x_i, we integrate the residual over the domain

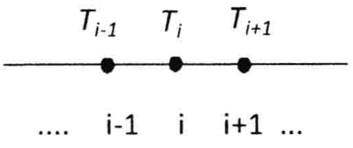
$$\int_0^1 \left(\frac{d^2\tilde{T}}{dx^2} + \tilde{T} + x \right) \delta(x - x_i)\, dx = 0 \quad \text{for} \quad i = 2,3...\, IL-1 \tag{15}$$

and there results the residual equation at the grid point x_i,

$$\left. \left(\frac{d^2\tilde{T}}{dx^2} + \tilde{T} + x \right) \right|_{x_i} = 0 \qquad \text{for} \quad i = 2,3...IL-1 \tag{16}$$

Using a local quadratic polynomial approximation for, $\tilde{T}(x)$, over grid points i – 1, i and i + 1, with the origin x = 0 located at the grid point x_i,

$$\tilde{T}(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 \tag{17}$$



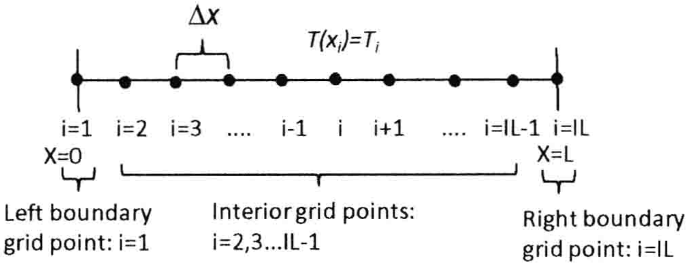


Figure 4. Discretization of the 1D domain used in the FDM.

one finds that

$$\tilde{T}(x) = T_i + \left(\frac{T_{i+1} - T_{i-1}}{2\Delta x} \right) x + \left(\frac{T_{i+1} - 2T_i + T_{i-1}}{2\Delta x^2} \right) x^2 \tag{18}$$

and upon introducing the above local approximation for the temperature into Eq. (16), we arrive at the interior FDM algebraic equation,

$$\frac{T_{i-1} - 2T_i + T_{i+1}}{\Delta x^2} + T_i + x_i = 0 \quad \text{for} \quad i = 2,3 \ldots IL-1 \tag{19}$$

that is re-arranged in the tri-diagonal form

$$\left(\frac{1}{\Delta x^2} \right) T_{i-1} + \left(1 - \frac{2}{\Delta x^2} \right) T_i + \left(\frac{1}{\Delta x^2} \right) T_{i+1} = -x_i \tag{20}$$

Defining the FDM coefficients, $a_i = \left(\frac{1}{\Delta x^2} \right), b_i = \left(1 - \frac{2}{\Delta x^2} \right), c_i = \left(\frac{1}{\Delta x^2} \right), d_i = -x_i$, and applying the first kind boundary conditions at $x = 0$ and $x = \text{L}$, the following set of tri-diagonal FDM equations is readily assembled and efficiently solved by the Thomas Algorithm,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ a_2 & b_2 & c_2 & 0 & \cdots & 0 \\ 0 & a_3 & b_3 & c_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{IL-1} & b_{IL-1} & c_{IL-1} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_{IL-1} \\ T_{IL} \end{Bmatrix} = \begin{Bmatrix} T_o \\ d_2 \\ d_3 \\ \vdots \\ d_{IL-1} \\ T_L \end{Bmatrix} \tag{21}$$

Using IL = 6 grid points, the MATHCAD spreadsheet calculation for the FDM is provided below:

Finite Difference Method:

$IL := 6$ ...no of grid points

$\Delta x := \frac{L}{IL - 1}$ $\quad \Delta x = 0.2$ .... mesh spacing

$x_i := (i - 1)\cdot \Delta x$ ... x-location of ith grid point

$a := \frac{1}{\Delta x^2}$ $\quad b := \left(1 - \frac{2}{\Delta x^2}\right)$ $\quad c := \frac{1}{\Delta x^2}$ ...FDM coefficients

$A_{i,j} := 0$

$d_i := 0$ ...initialize problem

Load FDM equations

$i := 1$ ...load left boundary condition

$A_{i,i} := 1$

$d_i := To$

$i := 2, 3 .. IL - 1$ ...load interior FDE's

$A_{i,i-1} := a$

$A_{i,i} := b$

$A_{i,i+1} := c$

$d_i := -x_i$

$i := IL$

$A_{i,i} := 1$ ...load right hand side temperature boundary conditions FDE

$d_i := TL$

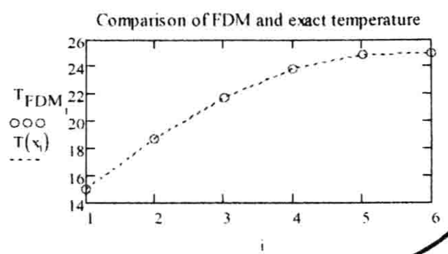
$i := 1, 2 .. IL$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 25 & -49 & 25 & 0 & 0 & 0 \\ 0 & 25 & -49 & 25 & 0 & 0 \\ 0 & 0 & 25 & -49 & 25 & 0 \\ 0 & 0 & 0 & 25 & -49 & 25 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad d = \begin{pmatrix} 15 \\ -0.2 \\ -0.4 \\ -0.6 \\ -0.8 \\ 25 \end{pmatrix}$$ ...echo FDM matrix and RHS

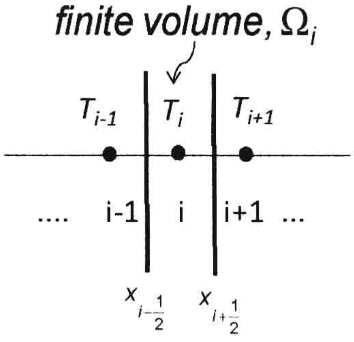
$T_{FDM} := A^{-1}\cdot d$ ...solve

$$err_{FDM_i} := \frac{\left|T_{FDM_i} - T(x_i)\right|}{\left|T(x_i)\right|}$$ ....compute relative error

$T_{FDM_i} =$	$T(x_i) =$	$err_{FDM_i} =$
15	15	0
18.73251	18.72608	$3.437\cdot 10^{-4}$
21.70772	21.69763	$4.653\cdot 10^{-4}$
23.79863	23.78822	$4.375\cdot 10^{-4}$
24.91359	24.90653	$2.833\cdot 10^{-4}$
25	25	0



Finite Volume Method – Subdomain MWR with Local Polynomial Trial Functions



In the finite volume method (FVM) [2, 3], the same discretization as in Fig. 4 may be used, except that now a subdomain (finite volume) $\Omega_i \in \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right]$ surrounding each grid point x_i is defined to extend from, $x_{i-\frac{1}{2}}$, the half way mark between i – 1 and i, and $x_{i+\frac{1}{2}}$ located at the half-way mark between i and i + 1. In this case the subdomain MWR is applied and

$$\int_0^1 \left(\frac{d^2\tilde{T}}{dx^2} + \tilde{T} + x\right) w_i(x)\,dx = 0 \qquad \text{for} \quad i=2,3...IL-1 \tag{22}$$

leads to

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\frac{d^2\tilde{T}}{dx^2} + \tilde{T} + x\right) dx = 0 \qquad \text{for} \quad i=2,3...IL-1 \tag{23}$$

since $w_i(x) = 0$ outside the subdomain Ω_i. Integrating the second derivative leads to

$$\left.\frac{d\tilde{T}}{dx}\right|_{x_{i+\frac{1}{2}}} - \left.\frac{d\tilde{T}}{dx}\right|_{x_{i-\frac{1}{2}}} + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(\tilde{T} + x\right) dx = 0 \qquad \text{for} \quad i=2,3...IL-1 \tag{24}$$

Noting that the first two terms are related to the flux in and out of the subdomain (finite volume) Ω_i, and this expression integrates the source term over the finite volume, unlike FDM that collocates and samples the generation term at the grid point, the FVM expresses a conservation principle on the grid. This is a distinction that becomes very important in non-linear and multi-dimensional problems. We are now left with introducing the approximation for $\tilde{T}(x)$ to arrive at the FVM algebraic analog. In FVM, various local interpolations are utilized. We shall use local linear interpolation between grid points to evaluate the $T(x)$ at the finite volume faces, so that,

$$\tilde{T}(x) = \alpha_1 + \alpha_2 x = \begin{cases} \dfrac{T_{i+1} + T_i}{2} + \left(\dfrac{T_{i+1} - T_i}{\Delta x}\right) x & \text{for} \quad x \in [\, x_i, x_{i+1}] \\[2ex] \dfrac{T_i + T_{i-1}}{2} + \left(\dfrac{T_i - T_{i-1}}{\Delta x}\right) x & \text{for} \quad x \in [\, x_{i-1}, x_i\,] \end{cases} \tag{25}$$

Resulting in the following expressions for the derivatives in Eq. (24),

$$\left.\frac{d\tilde{T}}{dx}\right|_{x_{i\pm\frac{1}{2}}} = \begin{cases} \left.\dfrac{d\tilde{T}}{dx}\right|_{x_{i+\frac{1}{2}}} = \left(\dfrac{T_{i+1} - T_i}{\Delta x}\right) \\[2ex] \left.\dfrac{d\tilde{T}}{dx}\right|_{x_{i-\frac{1}{2}}} = \left(\dfrac{T_i - T_{i-1}}{\Delta x}\right) \end{cases} \tag{26}$$

For consistency, using the trapezoidal rule that integrates a linear interpolation, and the interpolation we developed in Eq. (25), the integral of $\tilde{T}(x)$ over the finite volume is evaluated as

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \tilde{T}(x)dx = \left(\frac{\Delta x}{8}\right)T_{i-1} + \left(\frac{3\Delta x}{4}\right)T_i + \left(\frac{\Delta x}{8}\right)T_{i+1} \tag{27}$$

Integrating the source term analytically over of finite volume and putting it all together, Eq. (24) becomes

$$\left(\frac{T_{i+1}-T_i}{\Delta x}\right)-\left(\frac{T_i-T_{i-1}}{\Delta x}\right)+\left[\left(\frac{\Delta x}{8}\right)T_{i-1}+\left(\frac{3\Delta x}{4}\right)T_i+\left(\frac{\Delta x}{8}\right)T_{i+1}\right]=-\frac{1}{2}\left(x^2_{i+\frac{1}{2}}-x^2_{i-\frac{1}{2}}\right) \tag{28}$$

Diving by Δx, we arrive at the FVM algebraic analog,

$$\left(\frac{1}{\Delta x^2}+\frac{1}{8}\right)T_{i-1}+\left(\frac{3}{4}-\frac{2}{\Delta x^2}\right)T_i+\left(\frac{1}{\Delta x^2}+\frac{1}{8}\right)T_{i+1}=-\frac{1}{2}\left(x_{i+\frac{1}{2}}-x_{i-\frac{1}{2}}\right) \tag{29}$$

Defining the FVM coefficients, $a_i=\left(\frac{1}{\Delta x^2}+\frac{1}{8}\right), b_i=\left(\frac{3}{4}-\frac{2}{\Delta x^2}\right), c_i=\left(\frac{1}{\Delta x^2}+\frac{1}{8}\right), d_i= -\frac{1}{2}\left(x_{i+\frac{1}{2}}+x_{i-\frac{1}{2}}\right)$, and applying the first kind boundary conditions at x = 0 and x = L, we arrive at the same tri-diagonal form as in Eq. (21), except with different coefficients. Again using IL = 6 for consistency, the MATHCAD spreadsheet for the FVM implementation and its solution is provided:

Finite Volume Method:

$a := \left(\frac{1}{8} + \frac{1}{\Delta x^2}\right)$ $b := \left(\frac{3}{4} - \frac{2}{\Delta x^2}\right)$ $c := \left(\frac{1}{8} + \frac{1}{\Delta x^2}\right)$ ...FVM coefficients

$A_{i,j} := 0$

$d_i := 0$ ...initialize problem

Load FVM equations

$i := 1$ ...load left boundary condition

$A_{i,i} := 1$

$d_i := To$

$i := 2, 3.. IL - 1$ ...load interior FDE's

$A_{i,i-1} := a$

$A_{i,i} := b$

$A_{i,i+1} := c$

$d_i := -(i-1)\cdot\Delta x$

$i := IL$

$A_{i,i} := 1$ ...load right hand side temperature boundary conditions FDE

$d_i := TL$

$i := 1, 2.. IL$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 25.125 & -49.25 & 25.125 & 0 & 0 & 0 \\ 0 & 25.125 & -49.25 & 25.125 & 0 & 0 \\ 0 & 0 & 25.125 & -49.25 & 25.125 & 0 \\ 0 & 0 & 0 & 25.125 & -49.25 & 25.125 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad d = \begin{pmatrix} 15 \\ -0.2 \\ -0.4 \\ -0.6 \\ -0.8 \\ 25 \end{pmatrix}$$

$T_{FVM} := A^{-1}\cdot d$ ...solve

$err_{FVM_i} := \frac{\left|T_{FVM_i} - T(x_i)\right|}{\left|T(x_i)\right|}$ ....compute relative error

$T_{FVM_i} =$	$T(x_i) =$	$err_{FVM_i} =$
15	15	0
18.7229	18.72608	$1.699\cdot10^{-4}$
21.69264	21.69763	$2.299\cdot10^{-4}$
23.78308	23.78822	$2.162\cdot10^{-4}$
24.90304	24.90653	$1.4\cdot10^{-4}$
25	25	0