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身似别无长物
心如不系之舟
或问几何功业
一诺十年此书

Preface

Quantum integrable models play important roles in a variety of fields such as quantum field theory, condensed matter physics, and statistical physics. For decades, a number of theoretical methods have been developed for solving the eigenvalue problem of integrable models. Among them, the three typical and most popular methods are the coordinate Bethe Ansatz method proposed by H. Bethe in 1931, the $T-Q$ method proposed by R.J. Baxter in the early 1970s, and the algebraic Bethe Ansatz method proposed by the Leningrad Group in the late 1970s. These methods have been demonstrated as powerful in solving most of the known quantum integrable models. After Baxter's work on the eight-vertex model, people realized that a special class of quantum integrable models exists in which the $U(1)$ symmetry is broken and, in some cases, obvious reference states are absent. Some well-known examples are the *XYZ* spin chain (or equivalently the eight-vertex model), the quantum Toda chain, the anisotropic spin torus, and the quantum spin chains with nondiagonal boundary fields. Several methods have since been developed to approach this remarkable problem. Among them, two promising ones are Baxter's $T-Q$ method and Sklyanin's separation of variables (SoV) method, which provide efficient tools to treat quantum integrable models with functional analysis.

This book serves as an introduction to the off-diagonal Bethe Ansatz (ODBA) method, a newly developed analytic theory to approach exact solutions of quantum integrable models, especially those with nontrivial boundaries. In any sense, ODBA is not an isolated theory but one based on extensive existing knowledge. Therefore, this book also covers some main ingredients of $T-Q$ relation, algebraic Bethe Ansatz, thermodynamic Bethe Ansatz, fusion techniques and Sklyanin's SoV basis, etc. It is organized in a parallel structure to explain how ODBA works for different types of integrable models. Chapter 1 is devoted to the basic knowledge of quantum integrable models, and Chap. 2 to a comprehensive introduction of the algebraic Bethe Ansatz, the fusion techniques, and the SoV scheme. In addition, the thermodynamic Bethe Ansatz method is introduced as a tool for deriving the physical quantities. Chapter 3 focuses on the application of ODBA in the periodic *XXZ* model and the *XYZ* model, and Chap. 4 on the topological boundary problem using the anisotropic spin torus as example. Chapter 5 studies the exact solution of the spin-$\frac{1}{2}$ chain Hamiltonian with generic open boundaries, which had been a long-standing problem for over two decades. Chapter 6 is devoted to the one-dimensional Hubbard model and the super-symmetric $t-J$ model with generic integrable boundaries. Chapters 7 and 8 focus on the generalizations of ODBA to high-spin integrable models. Chapter 9 is devoted to the Izergin-Korepin model with generic boundaries, a typical integrable model beyond the A-type models. Calculations of some important physical quantities based on the Bethe Ansatz equations, especially the nontrivial boundary contributions, are given in Chaps. 2–5 and the method for retrieving the eigenstates based on the inhomogeneous $T-Q$ relations and the SoV basis is introduced with concrete examples in Chaps. 4 and 5.

In general, the authors aim to introduce topics that are under ongoing research and are developing at a stimulating pace in this fascinating field. These contents are selected for the book according to the authors' own understanding of the topics under discussion. Thus, they devote much attention to methods that work well for the nontrivial boundaries (Research on nontrivial surface effects, including edge states of the quantum Hall effect, surface states of topological insulators, open strings, and stochastic processes in nonequilibrium statistical physics, has become a trend in modern physics. The authors study this problem from the mathematical physics side.). The two-dimensional lattice models and most of the well-established knowledge on the models with periodic and diagonal boundary conditions are not included, since several excellent books have already covered these topics. This book was originally planned for around 100 pages but then was expanded to the present size, thanks to suggestions of numerous colleagues that detailed calculations should be included as much as possible to make it easy to follow the method. Although most of the results contained in this book have been rigorously proven, we still use the word "exactly" in the title as Baxter did for his book, for the reason that some results in this book are not that rigorously proven. For example, the thermodynamic limit is constructed based on reasonable physical arguments. For most models considered in this book, numerical results are provided to support the analytical ones, which is a conventional way for physicists and scientists in other fields to support their proposals, though it may not meet mathematical rigorousness. For physicists, to propose something correct is always more ambitious than to prove it.

Also, the methodology still needs to be developed and leaves some open questions, among which are: how to apply it in graded integrable models and in cyclic integrable models with nontrivial boundaries; how to retrieve the Bethe states and to derive the scalar products of high-rank quantum integrable models, etc. We expect that those issues may undergo significant progress in the near future.

The authors would like to share with you their happiness in undertaking the collaboration, which started in Fall 2012. At that time little was known about ODBA. Without the ensuing teamwork, it would have been impossible to achieve the main original results contained in this book!

We apologize if any important references are omitted. Any such error is definitely due to the limits of the authors' knowledge of the literature.

Acknowledgments Y. Wang would like to acknowledge a number of people who directly or indirectly helped to make the book possible. The first is F.C. Pu, who introduced the author into this interesting field. Under Pu's supervision, the author came to notice the problem of solving those integrable models without $U(1)$ symmetry 30 years ago when introduced to Baxter's papers and Takhtajan and Faddeev's paper about the XYZ model. As a junior graduate student, the author was unable to understand why the model could only be solved for an even number of sites. In 1997, when he became aware of the paper about the open *XYZ* model by H. Fan et al., he gradually realized that Baxter's local gauge transformation (vertex-face transformation or corner matrix) method could also be applied to the *XXZ* spin chain with nondiagonal boundary fields. The idea became clearer and clearer during his collaboration with J. Cao, H.-Q. Lin, and K. Shi. The author's thanks also go to D. Jin and L. Yu who have continually helped, supported, and encouraged him in his career. Fruitful discussions with R.J. Baxter and R.I. Nepomechie about the manuscript are especially acknowledged. Most importantly, the author would like to extend his deep gratitude to his wife Yan, who has been dedicating herself to taking care of the family so that her husband could devote more of his time to work even before the baby was coming!

W.-L. Yang, J. Cao, and K. Shi would like to express deep thanks to their supervisor B.-Y. Hou, whose strict style of scholarship has influenced them a lot in their career. They also benefited much from the academic atmosphere of Institute of Modern Physics of Northwest University founded by B.-Y. Hou.

All the authors would like to acknowledge some of the referees for their constructive comments on several of the authors' original papers. Remarkably, a question of "how to get the root distribution of the Bethe Ansatz equations" stimulated the authors to write the paper on the thermodynamic limit, while the comments of "how to prove the completeness of the solutions" and "what is the corresponding eigenstate in the homogeneous lattice case" stimulated the authors to propose the two theorems in Chap. 1 and related corollaries and to retrieve Bethe states based on the inhomogeneous $T-Q$ relations. These papers become important parts of the ODBA scheme.

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Beijing, February 2015	Yupeng Wang
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Contents

Chapter 1
Overview

Quantum integrable models are exactly solvable models defined by the Yang-Baxter equation (YBE) [1, 2] or the Lax representation [3]. These models play important roles in a variety of fields such as quantum field theory, condensed matter physics and statistical physics, because they can provide solid benchmarks for understanding the many-body effects in corresponding universal classes and sometimes even yield conclusions to debates about important physical concepts. For instance, the exact solution of the two-dimensional Ising model [4] gives concrete evidence for the existence of thermodynamic phase transitions; the exact solution of the one-dimensional Hubbard model [5] clarifies the concept of the Mott insulator; while the spinon excitations obtained from the exact solution of the Heisenberg spin chain [6] elucidate how fractional charges could be generated from low-dimensional correlated quantum systems. In recent years, new applications have been found in cold atom systems, quantum information, AdS/CFT correspondence and many other aspects. For example, the Lieb-Liniger model [7, 8], the δ-potential Fermi gas model [1, 9] and the one-dimensional Hubbard model [5] have provided important benchmarks for one-dimensional cold atom systems and even fit experimental data with incredibly high accuracy [10]. On the other hand, the anomalous dimensions of operators of $\mathcal{N} = 4$ super-symmetric Yang-Mills field theory can be given by the eigenvalues of the Hamiltonians for certain integrable spin chains [11, 12].

For several decades, a number of theoretical methods have been proposed for solving the eigenvalue problem of quantum integrable models. Among them, the three typical and most popular methods are the coordinate Bethe Ansatz method proposed by Bethe [13], the $T - Q$ method proposed by Baxter [14, 15] and the algebraic Bethe Ansatz method [16–22] proposed by the Leningrad Group. Those methods have been demonstrated to be powerful in solving the eigenvalue problem of the known quantum integrable models and a great number of papers have been devoted to this topic in the literature. Among the family of quantum integrable models, there exists a large class of models that do not possess $U(1)$ symmetry and an obvious reference state is usually absent. Some well-known examples are the *XYZ* spin chain with an odd number of sites [17], the anisotropic spin torus [23] and the quantum spin chains with non-diagonal boundary fields [24–27]. These models have been

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found to possess important applications in non-equilibrium statistical physics (e.g., stochastic processes [28–33]), in condensed matter physics (e.g., a Josephson junction embedded in a Luttinger liquid [34], spin-orbit coupling systems, one-dimensional cold atoms coupled with a BEC reservoir, etc.) and in high energy physics (e.g., open strings and coupled D-Branes). Many efforts have been made [24–27, 29–32, 35–53] to approach this nontrivial problem.

Actually, Baxter's theory [2] already provided a powerful method for approaching exactly solvable models with functional analysis, allowing us to solve those models without $U(1)$ symmetry. A remarkable example is the exact solution of the eight-vertex model [2]. Another important functional analysis method to approach these models is the quantum separation of variables (SoV) method [54–57] proposed by Sklyanin, which has also been successfully applied to several nontrivial quantum integrable models. A famous example is the solution of the quantum Toda chain [54]. Nevertheless, for a long time, the Bethe Ansatz equations could only be obtained for constrained boundary conditions [24, 25, 37] or for special crossing parameters [26, 27, 35, 36] associated with spin-$\frac{1}{2}$ chains or with spin-s chains [58–61]. An analytic method for solving the integrable models with or without obvious reference state, i.e., the off-diagonal Bethe Ansatz (ODBA) method was proposed in 2013 [62]. With this method, several models without obvious reference states were solved exactly [62–71] by the construction of the inhomogeneous $T - Q$ relations, and a method to obtain the physical quantities in the thermodynamic limit was established [72] subsequently, based on the ODBA equations. Soon after that, Sklyanin's SoV method was applied to the spin-$\frac{1}{2}$ chains with generic integrable boundaries [73], and a set of Bethe states was conjectured via the algebraic Bethe Ansatz [74]. A systematic method to retrieve the Bethe-type eigenstates based on the ODBA solutions and the SoV basis is developed in [75, 76].

This chapter is a brief introduction of the integrability associated with YBE; the boundary conditions associated with the integrability; the factorizability induced by YBE and the ideas of the coordinate Bethe Ansatz; the $T - Q$ relation; and the basic ingredients of ODBA.

1.1 Integrability and Yang-Baxter Equation

The concept of integrability originated from classical mechanics, wherein a physical system is usually described by a set of differential equations (the equations of motion). The solutions of these differential equations are their integrals. In such a sense, integrable means solvable. The integrals are accompanied by some integral constants that do not depend on time and are usually called integrals of motion or conserved quantities. For a mechanical system with N degrees of freedom, if N independent integrals of motion which are in involution can be obtained, then the system is completely integrable.

A precise definition of classical integrability is given by Liouville's theorem [77]: Given a Hamiltonian system described by the coordinates $\{x_j | j = 1, \ldots, N\}$ and the momenta $\{k_j | j = 1, \ldots, N\}$, if there exists a canonical transformation $x_j, k_j \to q_j, p_j$ to make the Hamiltonian to be only a function of the canonical momenta $\{p_j\}$, the system is integrable. This is true because the following Poisson brackets hold:

$$\begin{aligned} &\{k_j, x_l\} = \delta_{j,l}, \quad \{p_j, q_l\} = \delta_{j,l}, \\ &\frac{dp_j}{dt} = \{H, p_j\} = 0, \quad \frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \end{aligned} \tag{1.1.1}$$

which imply that the N canonical momenta are conserved quantities and the evolution of the N canonical coordinates is linear in time t. The Liouville's theorem indicates that the variables of an integrable system are in fact completely separable. However, such a separation process is usually rather nontrivial.

To show the integrability clearly, let us first consider a simple classical integrable system which might give a bridge to the quantum integrable systems: N classical indistinguishable objects moving in a straight line. Suppose each object carries a momentum k_j initially and the collisions among the objects are elastic. Consider the collision process between two neighboring objects. If the objects carry momenta k_i and k_j before the collision, and k_i' and k_j' after the collision, respectively, the conservation laws of momentum and energy require that

$$k_i + k_j = k_i' + k_j', \tag{1.1.2}$$

$$k_i^2 + k_j^2 = k_i'^2 + k_j'^2. \tag{1.1.3}$$

The above equations have two sets of solutions: (1) $k_i' = k_i, k_j' = k_j$; (2) $k_i' = k_j$, $k_j' = k_i$. Since these objects are not penetrable, only the second set of solutions is allowed, i.e., the objects exchange their momenta after the collision. If the objects are moving in a ring (periodic boundary condition), the system can be described by a parameter set $\{k_1, \ldots, k_N\}$ which does not change with the collision processes. Such phenomenon is usually called non-diffraction behavior and is a common feature of the integrable systems. Obviously, the following conserved quantities hold:

$$\mathrm{C}_n = \sum_{j=1}^{N} k_j^n, \quad n = 1, \ldots, N, \tag{1.1.4}$$

indicating that this system is completely integrable. If the objects move in an interval with boundaries, the momenta they carry are no longer a conserved set of parameters. The object carrying a momentum k_j must be reflected at the boundaries and its momentum is changed to $-k_j$ after reflection. However, the system still preserves its integrability because of the existence of the following conserved quantities:

$$\mathrm{C}_n^o = \sum_{j=1}^{N} |k_j|^n, \quad n = 1, \ldots, N. \tag{1.1.5}$$

The central point of quantum integrability lies in the conservation laws governed by the YBE. There are several ways to derive YBE. Here we adopt Yang's procedure [1]. Consider that N indistinguishable quantum particles are moving in one spatial dimension. Suppose its wave function initially takes the following asymptotic form:

$$\Psi_{in} \sim e^{i\sum_{j=1}^{N} k_j x_j}, \quad x_1 \ll x_2 \ll \cdots \ll x_N. \tag{1.1.6}$$

The first particle reaches the right end of the system from the left after scattering with all the other particles. The asymptotic wave function after this process becomes

$$\Psi_{out} \sim S_{1,23\ldots N} e^{i\sum_{j=1}^{N} k_j x_j}, \quad x_2 \ll x_3 \ll \cdots \ll x_N \ll x_1, \tag{1.1.7}$$

where $S_{1,23\ldots N}$ is the scattering matrix of particle 1 to all the other particles. If the many-body S-matrix can be factorized as the product of two-body S-matrices

$$S_{1,23\ldots N} = S_{1,N}(k_1, k_N) \cdots S_{1,3}(k_1, k_3) S_{1,2}(k_1, k_2), \tag{1.1.8}$$

we call the system a factorizable system. We note that the following inversion identity for the two-body S-matrix holds:

$$S_{1,2}(k_1, k_2) S_{2,1}(k_2, k_1) = 1. \tag{1.1.9}$$

The factorizability ensures the integrability of a quantum system. To show this point clearly, let us consider the three-particle case. There are two routes from the initial state $|1, 2, 3\rangle$ to the final state $|3, 2, 1\rangle$ as shown in Fig. 1.1. These two routes must be equivalent because of the uniqueness of the final wave function. If the system is factorizable, we have the following equation:

$$S_{1,2}(k_1, k_2) S_{1,3}(k_1, k_3) S_{2,3}(k_2, k_3) = S_{2,3}(k_2, k_3) S_{1,3}(k_1, k_3) S_{1,2}(k_1, k_2). \tag{1.1.10}$$

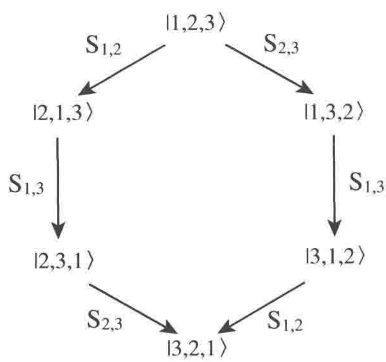

Fig. 1.1 Schematic diagram of the Yang-Baxter equation: the two routes from the initial state $|1, 2, 3\rangle$ to the final state $|3, 2, 1\rangle$ must be equivalent

This is the YBE, which was realized in [78] and first emphasized by Yang [1] in solving the one-dimensional δ-potential Fermi gas model and by Baxter [2] in constructing the $T-Q$ method for solving the two-dimensional vertex models. It was demonstrated by Yang [79] that YBE is the sufficient condition of Yang-Baxter quantum integrability with proper boundary conditions. This equation also ensures factorizability, thus constituting the cornerstone for constructing and solving the quantum integrable models. In fact, the factorizability indicates that the basic scattering process is the two-body one, and that some conserved quantities that possess the eigenvalues of Eqs. (1.1.4) or (1.1.5) always exist, because of the conservation laws of momentum and energy.

Usually, if the spectral parameters in the S-matrix are additive, i.e., $S_{i,j}(k_i, k_j) \sim R_{i,j}(k_i - k_j)$, the YBE is written as

$$\begin{aligned} R_{i,j}(u_i - u_j)R_{i,k}(u_i - u_k)R_{j,k}(u_j - u_k) \\ = R_{j,k}(u_j - u_k)R_{i,k}(u_i - u_k)R_{i,j}(u_i - u_j). \end{aligned} \quad (1.1.11)$$

Throughout this book we adopt the standard notations: for any matrix $O \in \mathrm{End}(\mathbf{V})$, O_j is an embedding operator in the tensor space $\mathbf{V} \otimes \mathbf{V} \otimes \cdots \otimes \mathbf{V}$, which acts as O on the jth factor space and as identity on the other factor spaces; $R_{i,j}(u)$ is an embedding operator of the R-matrix in the tensor space, which acts as identity on the factor spaces except for the ith and jth ones. Moreover, we denote id as the identity operator in the corresponding space.

1.2 Integrable Boundary Conditions

There are several possible boundary conditions associated with the quantum integrability. To show them clearly, let us first introduce the procedure for constructing quantum integrable models based on YBE. In principle, given an R-matrix, we can seek solutions of the equation

$$R_{0,\bar{0}}(u - v)L_{0,n}(u)L_{\bar{0},n}(v) = L_{\bar{0},n}(v)L_{0,n}(u)R_{0,\bar{0}}(u - v). \quad (1.2.1)$$

Obviously, $L_{0,n}(u) = R_{0,n}(u - \theta_n)$ is a solution of this equation. $L_{0,n}(u)$ is usually called the Lax operator and θ_n is a site-dependent parameter (inhomogeneous parameter). Given an R-matrix satisfying YBE, we define the monodromy matrix

$$T_0(u) = L_{0,N}(u)L_{0,N-1}(u) \cdots L_{0,1}(u), \quad (1.2.2)$$

where N is the number of sites of the system. The transfer matrix of the system is defined as the trace of the corresponding monodromy matrix in the auxiliary space

$$t(u) = tr_0 T_0(u). \tag{1.2.3}$$

The concept of the transfer matrix originated from the classical statistical models [2] and was adopted later in the study of quantum integrable models.

An important step to construct and to solve quantum integrable models is the RTT relation proposed by Baxter. Since

$$[L_{0,m}(u), L_{\bar{0},n}(v)] = 0, \quad m \neq n, \tag{1.2.4}$$

from YBE (1.2.1) we have

$$\begin{aligned} &R_{0,\bar{0}}(u-v)T_0(u)T_{\bar{0}}(v) \\ &\quad = R_{0,\bar{0}}(u-v)L_{0,N}(u)L_{\bar{0},N}(v)\cdots L_{0,1}(u)L_{\bar{0},1}(v) \\ &\quad = L_{\bar{0},N}(v)L_{0,N}(u)R_{0,\bar{0}}(u-v)\cdots L_{0,1}(u)L_{\bar{0},1}(v) \\ &\quad = L_{\bar{0},N}(v)L_{0,N}(u)\cdots L_{\bar{0},1}(v)L_{0,1}(u)R_{0,\bar{0}}(u-v) \\ &\quad = T_{\bar{0}}(v)T_0(u)R_{0,\bar{0}}(u-v). \end{aligned} \tag{1.2.5}$$

Multiplying $R^{-1}_{0,\bar{0}}(u-v)$ from the left side of the Eq. (1.2.5) and taking the trace in the auxiliary spaces 0 and $\bar{0}$, we obtain

$$[t(u),\ t(v)] = 0. \tag{1.2.6}$$

Expanding $t(u)$ in terms of u

$$t(u) = \sum_{n=0}^{\infty} t^{(n)} u^n, \tag{1.2.7}$$

we readily have that the coefficients are mutually commuting

$$[t^{(m)}, t^{(n)}] = 0. \tag{1.2.8}$$

Choosing one of them or a certain combination of them as a Hamiltonian H, then $[H, t^{(n)}] = 0$ and the model is integrable. If we obtain the eigenvalues of the transfer matrix, we can obtain all the eigenvalues of the coefficients. The boundary condition for the transfer matrix defined by (1.2.2) and (1.2.3) is periodic.

In most of the cases, Eq. (1.2.1) allows c-number solution $L_{0,n}(u) = G_0$ which is independent of the spectral parameter u. This allows us to construct the following transfer matrix

$$t(u) = tr_0\{G_0 L_{0,N}(u) L_{0,N-1}(u) \cdots L_{0,1}(u)\}, \tag{1.2.9}$$