TURING 图灵原版数学·统计学系列 21

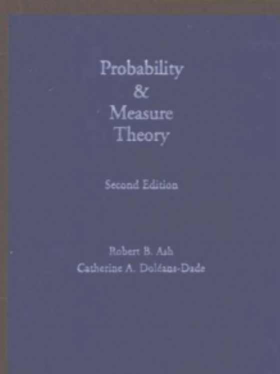
ELSEVIER 爱思唯尔

Probability & Measure Theory

概率与测度论

（英文版·第2版）

[美] Robert B. Ash
Catherine A. Doléans-Dade 著



人民邮电出版社
POSTS & TELECOM PRESS

Probability & Measure Theory

概率与测度论 （英文版·第2版）

“本书非常漂亮地从概率的角度来阐述测度理论。……本书的一大优点是在最后给出了部分习题的完全解答，这给了自学的读者极大的帮助。”

——Guy Lebanon，卡内基·梅隆大学

“如果您是一位懂得基础概率的金融工作者，还想更深入地学习概率方面的知识，那么这本书正是您需要的。本书行文简洁，易于理解，选取的课题非常适合金融工程和金融风险管理人员学习参考。……”

——http://www.contingencyanalysis.com 网站评论

概率论作为一门基础数学分支，因其严密的逻辑性和完美的表达形式在物理、医学、金融等学科及应用领域发挥着重要作用。

本书是概率论和测度论名著，行文流畅，主线清晰。全书各节都附有习题，而且在书后提供了部分习题的详细解答。本书可作为相关专业高年级本科生或研究生的教材，也可供相关专业的高校师生和科研人员参考。

Robert B. Ash 伊利诺伊大学数学系教授。世界著名数学家，研究领域包括：信息理论、代数、拓扑、概率论、泛函分析等。主要著作有*Measure*、*Integration and Functional Analysis*和*Information Theory* 等。

Catherine A. Doléans-Dade 已故世界著名数学家。生前曾长期担任伊利诺伊大学–厄巴纳香槟分校数学系概率组成员，在随机分析领域有诸多建树。她在鞅论中提出的Doléans测度已广为人知。

ELSEVIER 爱思唯尔

本书相关信息请访问：图灵网站 http://www.turingbook.com
读者/作者热线：(010)88593802
反馈/投稿/推荐信箱：contact@turingbook.com

分类建议 数学/概率

人民邮电出版社网址 www.ptpress.com.cn

ISBN 978-7-115-16615-9

9 787115 166159 >

ISBN 978-7-115-16615-9/O1

定价：69.00 元

TURING 图灵原版数学·统计学系列

Probability & Measure Theory

概率与测度论

（英文版·第2版）

[美] Robert B. Ash
Catherine A. Doléans-Dade 著

人 民 邮 电 出 版 社
北 京

图书在版编目（CIP）数据

概率与测度论：第2版：英文／（美）阿什（Ash. R. B.）（美）多朗-戴德(Doléans-Dade, C. A.）著．—北京：人民邮电出版社，2007.10
（图灵原版数学·统计学系列）

ISBN 978-7-115-16615-9

I. 概… II. ①阿… ②多… III. ①概率论－教材－英文 ②测度（数学）－教材－英文 IV. O211 O174.12

中国版本图书馆CIP数据核字（2007）第113707号

内容提要

本书是测度论和概率论领域的名著，行文流畅，主线清晰，材料取舍适当，内容包括测度和积分论、泛函分析、条件概率和期望、强大数定理和鞅论、中心极限定理、遍历定理以及布朗运动和随机积分等．全书各节都附有习题，而且在书后提供了大部分习题的详细解答．

本书可作为相关专业高年级本科生或研究生的双语教材，适合作为一学年的教学内容．也可选用其中部分章节用作一学期的教学内容或参考书．

图灵原版数学•统计学系列

概率与测度论（英文版•第2版）

◆ 著 ［美］Robert B. Ash Catherine A. Doléans-Dade
责任编辑 明永玲

◆ 人民邮电出版社出版发行 北京市崇文区夕照寺街14号
邮编 100061 电子函件 315@ptpress.com.cn
网址 http://www.ptpress.com.cn
北京铭成印刷有限公司印刷
新华书店总店北京发行所经销

◆ 开本：700×1000 1/16
印张：33.25
字数：549千字 2007年10月第1版
印数：1–3 000册 2007年10月北京第1次印刷

著作权合同登记号 图字：01-2007-3371号

ISBN 978-7-115-16615-9/O1

定价：69.00元

读者服务热线：(010)88593802 印装质量热线：(010)67129223

版 权 声 明

Probability & Measure Theory, Second Edition by Robert B. Ash, Catherine A. Doléans-Dade, ISBN: 978-0-12-065202-0.

Authorized English language reprint edition published by the Proprietor.

ISBN: 978-981-259-948-3.

Elsevier (Singapore) Pte Ltd.

3 Killiney Road

#08-01 Winsland House I

Singapore 239519

Tel：(65)6349-0200

Fax：(65)6733-1817

First Published 2007

2007年初版

Preface

It is a pleasure to accept the invitation of Harcourt/Academic Press to publish a second edition. The first edition has been used mainly in graduate courses in measure and probability, offered by departments of mathematics and statistics and frequently taken by engineers. We have prepared the present text with this audience in mind, and the title has been changed from *Real Analysis and Probability* to *Probability and Measure Theory* to reflect the revisions we have made.

Chapters 1 and 2 develop the fundamentals of measure and integration theory. Included are several results that are crucial in constructing the foundations of probability: the Radon–Nikodym theorem, the product measure theorem, the Kolmogorov extension theorem and the theory of weak convergence of measures. We remain convinced that it is best to assemble a complete set of measure-theoretic tools before going into probability, rather than try to develop both areas simultaneously. The gain in efficiency far outweighs any temporary loss in motivation. Those who wish to reach probability as quickly as possible may omit Chapter 3, which gives a brief introduction to functional analysis, and Section 2.3, which gives some applications to real analysis. In addition, instructors may wish to summarize or sketch some of the intricate constructions in Sections 1.3, 1.4, and 2.7.

The study of probability begins with Chapter 4, which offers a summary of an undergraduate probability course from a measure–theoretic point of view. Chapter 5 is concerned with the general concept of conditional probability and expectation. The approach to problems that involve conditioning, given events of probability zero, is the gateway to many areas of probability theory. Chapter 6 deals with strong laws of large numbers, first from the classical viewpoint, and then via martingale theory. Basic properties and applications of martingale sequences are developed systematically. Chapter 7 considers the central limit problem, emphasizing the fundamental role of Prokhorov's weak compactness theorem. The last two sections of this chapter cover some material (not in the first edition) of special interest to statisticians: Slutsky's theorem, the Skorokhod construction, convergence of transformed sequences and a k-dimensional central limit theorem.

Chapters 8 and 9 have been added in the second edition, and should be of interest to the entire prospective audience: mathematicians, statisticians, and engineers. Chapter 8 covers ergodic theory, which is developed far enough so that connections with information theory are clearly visible. The Shannon–McMillan theorem is proved and the isomorphism problem for Bernoulli shifts is discussed. Chapter 9 treats the one-dimensional Brownian motion in detail, and then introduces stochastic integrals and the Itô differentiation formula.

To make room for the new material, the appendix on general topology and the old Chapter 4 on the interplay between measure theory and topology have been removed, along with the section on topological vector spaces in Chapter 3. We assume that the reader has had a course in basic analysis and is familiar with metric spaces, but not with general topology. All the necessary background appears in *Real Variables With Basic Metric Space Topology* by Robert B. Ash, IEEE Press, 1993. (The few exercises that require additional background are marked with an asterisk.)

It is theoretically possible to read the text without any prior exposure to probability, picking up the necessary equipment in Chapter 4. But we expect that in practice, almost all readers will have taken a standard undergraduate probability course. We believe that discrete time, discrete state Markov chains, and random walks are best covered in a second undergraduate probability course, without measure theory. But instructors and students usually find this area appealing, and we discuss the symmetric random walk on $\mathbb{R}^k$ in Appendix 1.

Problems are given at the end of each section. Fairly detailed solutions are given to many problems, and instructors may obtain solutions to those problems in Chapters 1–8 not worked out in the text by writing to the publisher.

Catherine Doleans–Dade wrote Chapter 9, and offered valuable advice and criticism for the other chapters. Mel Gardner kindly allowed some material from *Topics in Stochastic Processes* by Ash and Gardner to be used in Chapter 8. We appreciate the encouragement and support provided by the staff at Harcourt/Academic Press.

Robert B. Ash
Catherine Doleans–Dade
Urbana, Illinois, 1999

Summary of Notation

We indicate here the notational conventions to be used throughout the book. The numbering system is standard; for example, 2.7.4 means Chapter 2, Section 7, Part 4. In the appendices, the letter A is used; thus A2.3 means Part 3 of Appendix 2.

The symbol $\square$ is used to mark the end of a proof.

1 Sets

If A and B are subsets of a set Ω, $A \cup B$ will denote the *union* of A and B, and $A \cap B$ the *intersection* of A and B. The union and intersection of a family of sets A_i are denoted by $\bigcup_i A_i$ and $\bigcap_i A_i$. The *complement* of A (relative to Ω) is denoted by A^c.

The statement "B is a *subset* of A" is denoted by $B \subset A$; the inclusion need not be proper, that is, we have $A \subset A$ for any set A. We also write $B \subset A$ as $A \supset B$, to be read "A is an *overset* (or *superset*) of B."

The notation $A - B$ will always mean, unless otherwise specified, the set of points that belong to A but not to B. It is referred to as the *difference* between A and B; a *proper difference* is a set $A - B$, where $B \subset A$.

The *symmetric difference* between A and B is by definition the union of $A - B$ and $B - A$; it is denoted by $A \,\Delta\, B$.

If $A_1 \subset A_2 \subset \cdots$ and $\bigcup_{n=1}^{\infty} A_n = A$, we say that the A_n form an *increasing* sequence of sets (increasing to A) and write $A_n \uparrow A$. Similarly, if $A_1 \supset A_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} A_n = A$, we say that the A_n form a *decreasing* sequence of sets (decreasing to A) and write $A_n \downarrow A$.

The word "includes" will always imply a subset relation, and the word "contains" a membership relation. Thus if $\mathscr{C}$ and $\mathscr{D}$ are collections of sets, "$\mathscr{C}$ includes $\mathscr{D}$" means that $\mathscr{D} \subset \mathscr{C}$. Equivalently, we may say that $\mathscr{C}$ contains all sets in $\mathscr{D}$, in other words, each $A \in \mathscr{D}$ is also a member of $\mathscr{C}$.

A *countable* set is one that is either finite or countably infinite.

The *empty set* $\emptyset$ is the set with no members. The sets A_i, $i \in I$, are *disjoint* if $A_i \cap A_j = \emptyset$ for all $i \neq j$.

2 Real Numbers

The set of real numbers will be denoted by $\mathbb{R}$, and $\mathbb{R}^n$ will denote n-dimensional Euclidean space. In $\mathbb{R}$, the interval $(a, b]$ is defined as $\{x \in \mathbb{R}\colon a < x \le b\}$, and (a, ∞) as $\{x \in \mathbb{R}\colon x > a\}$; other types of intervals are defined similarly. If $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are points in $\mathbb{R}^n$, $a \le b$ will mean $a_i \le b_i$ for all i. The interval $(a, b]$ is defined as $\{x \in R^n\colon a_i < x_i \le b_i, i = 1, \dots, n\}$, and other types of intervals are defined similarly.

The set of *extended real numbers* is the two-point compactification $\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$, denoted by $\overline{\mathbb{R}}$; the set of n-tuples $(x_1, \dots, x_n)$, with each $x_i \in \overline{\mathbb{R}}$, is denoted by $\overline{\mathbb{R}}^n$. We adopt the following rules of arithmetic in $\overline{\mathbb{R}}$:

$$a + \infty = \infty + a = \infty, \qquad a - \infty = -\infty + a = -\infty, \qquad a \in \mathbb{R},$$

$$\infty + \infty = \infty, \qquad -\infty - \infty = -\infty \qquad (\infty - \infty \text{ is not defined}),$$

$$b \cdot \infty = \infty \cdot b = \begin{cases} \infty & \text{if} \quad b \in \overline{\mathbb{R}} \quad b > 0, \\ -\infty & \text{if} \quad b \in \overline{\mathbb{R}}, \quad b < 0, \end{cases}$$

$$\frac{a}{\infty} = \frac{a}{-\infty} = 0, \quad a \in \mathbb{R} \quad \left(\frac{\infty}{\infty} \text{ is not defined}\right),$$

$$0 \cdot \infty = \infty \cdot 0 = 0.$$

The rules are convenient when developing the properties of the abstract Lebesgue integral, but it should be emphasized that $\overline{\mathbb{R}}$ is not a field under these operations.

Unless otherwise specified, *positive* means (strictly) greater than zero, and *nonnegative* means greater than or equal to zero.

The set of *complex numbers* is denoted by $\mathbb{C}$, and the set of n-tuples of complex numbers by $\mathbb{C}^n$.

3 Functions

If f is a function from Ω to Ω' (written as $f\colon \Omega \to \Omega'$) and $B \subset \Omega'$, the *preimage* of B under f is given by $f^{-1}(B) = \{\omega \in \Omega\colon f(\omega) \in B\}$. It follows from the definition that $f^{-1}(\bigcup_i B_i) = \bigcup_i f^{-1}(B_i)$, $f^{-1}(\bigcap_i B_i) = \bigcap_i f^{-1}(B_i)$, $f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$; hence $f^{-1}(A^c) = [f^{-1}(A)]^c$. If $\mathscr{C}$ is a class of sets, $f^{-1}(\mathscr{C})$ means the collection of sets $f^{-1}(B)$, $B \in \mathscr{C}$.

If $f\colon \mathbb{R} \to \mathbb{R}$, f is *increasing* iff $x < y$ implies $f(x) \le f(y)$; *decreasing* iff $x < y$ implies $f(x) \ge f(y)$. Thus, "increasing" and "decreasing" do not have the strict connotation. If $f_n\colon \Omega \to \overline{\mathbb{R}}$, $n = 1, 2, \dots$, the f_n are said to form an *increasing sequence* iff $f_n(\omega) \le f_{n+1}(\omega)$ for all n and ω; a *decreasing sequence* is defined similarly.

If f and g are functions from Ω to $\overline{\mathbb{R}}$, statements such as $f \leq g$ are always interpreted as holding pointwise, that is, $f(\omega) \leq g(\omega)$ for all $\omega \in \Omega$. Similarly, if f_i: $\Omega \to \overline{\mathbb{R}}$ for each $i \in I$, $\sup_i f_i$ is the function whose value at ω is $\sup\{f_i(\omega):\ i \in I\}$.

If $f_1, f_2, \ldots$ form an increasing sequence of functions with limit f [that is, $\lim_{n\to\infty} f_n(\omega) = f(\omega)$ for all ω], we write $f_n \uparrow f$. (Similarly, $f_n \downarrow f$ is used for a decreasing sequence.)

Sometimes, a set such as $\{\omega \in \Omega:\ f(\omega) \leq g(\omega)\}$ is abbreviated as $\{f \leq g\}$; similarly, the preimage $\{\omega \in \Omega:\ f(\omega) \in B\}$ is written as $\{f \in B\}$.

If $A \subset \Omega$, the *indicator* of A is the function defined by $I_A(\omega) = 1$ if $\omega \in A$ and by $I_A(\omega) = 0$ if $\omega \notin A$. The phrase "characteristic function" is often used in the literature, but we shall not adopt this term here.

If f is a function of two variables x and y, the symbol $f(x, \cdot)$ is used for the mapping $y \to f(x, y)$ with x fixed.

The *composition* of two functions X: $\Omega \to \Omega'$ and f: $\Omega' \to \Omega''$ is denoted by $f \circ X$ or $f(X)$.

If f: $\Omega \to \overline{\mathbb{R}}$, the *positive* and *negative parts* of f are defined by $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$, that is,

$$f^+(\omega) = \begin{cases} f(\omega) & \text{if} \quad f(\omega) \geq 0, \\ 0 & \text{if} \quad f(\omega) < 0, \end{cases}$$

$$f^-(\omega) = \begin{cases} -f(\omega) & \text{if} \quad f(\omega) \leq 0, \\ 0 & \text{if} \quad f(\omega) > 0. \end{cases}$$

4 Topology

A *metric space* is a set Ω with a function d (called a *metric*) from $\Omega \times \Omega$ to the nonnegative reals, satisfying $d(x, y) \geq 0$, $d(x, y) = 0$ iff $x = y$, $d(x, y) = d(y, x)$, and $d(x, z) \leq d(x, y) + d(y, z)$. If $d(x, y)$ can be 0 for $x \neq y$, but d satisfies the remaining properties, d is called a *pseudometric* (the term *semimetric* is also used in the literature).

A *ball* (or *open ball*) in a metric or pseudometric space is a set of the form $B(x, r) = \{y \in \Omega:\ d(x, y) < r\}$ where x, the *center* of the ball, is a point of Ω, and r, the *radius*, is a positive real number. A *closed ball* is a set of the form $\overline{B}(x, r) = \{y \in \Omega:\ d(x, y) \leq r\}$.

Sequences in Ω are denoted by $\{x_n, n = 1, 2, \ldots\}$. The term "lower semicontinuous" is abbreviated LSC, and "upper semicontinuous" is abbreviated USC.

No knowledge of general topology (beyond metric spaces) is assumed, and the few comments that refer to general topological spaces can safely be ignored.

5 Vector Spaces

The terms "vector space" and "linear space" are synonymous. All vector spaces are over the real or complex field, and the complex field is assumed unless the term "real vector space" is used.

A *Hamel basis* for a vector space L is a maximal linearly independent subset B of L. (Linear independence means that if $x_1, \ldots, x_n \in B$, $n = 1, 2, \ldots$, and $c_1, \ldots, c_n$ are scalars, then $\sum_{i=1}^{n} c_i x_i = 0$ iff all $c_i = 0$.) Alternatively, a Hamel basis is a linearly independent subset B with the property that each $x \in L$ is a finite linear combination of elements in B. [An *orthonormal basis* for a Hilbert space (Chapter 3) is a different concept.]

The terms "subspace" and "linear manifold" are synonymous, each referring to a subset M of a vector space L that is itself a vector space under the operations of addition and scalar multiplication in L. If there is a metric on L and M is a closed subset of L, then M is called a *closed subspace.*

If B is an arbitrary subset of L, the *linear manifold generated by* B, denoted by $L(B)$, is the smallest linear manifold containing all elements of B, that is, the collection of finite linear combinations of elements of B. Assuming a metric on L, the *space spanned by* B, denoted by $S(B)$, is the smallest closed subspace containing all elements of B. Explicitly, $S(B)$ is the closure of $L(B)$.

6 Zorn's Lemma

A *partial ordering* on a set S is a relation "$\leq$" that is

(1) *reflexive*: $a \leq a$;
(2) *antisymmetric*: if $a \leq b$ and $b \leq a$, then $a = b$; and
(3) *transitive*: if $a \leq b$ and $b \leq c$, then $a \leq c$.

(All elements a, b, c belong to S.)

If $C \subset S$, C is said to be *totally ordered* iff for all $a, b \in C$, either $a \leq b$ or $b \leq a$. A totally ordered subset of S is also called a *chain* in S.

The form of Zorn's lemma that will be used in the text is as follows.

Let S be a set with a partial ordering "$\leq$." Assume that every chain C in S has an upper bound; in other words, there is an element $x \in S$ such that $x \geq a$ for all $a \in C$. Then S has a maximal element, that is, an element m such that for each $a \in S$ it is not possible to have $m \leq a$ and $m \neq a$.

Zorn's lemma is actually an axiom of set theory, equivalent to the axiom of choice.

Contents

CHAPTER

1

FUNDAMENTALS OF MEASURE AND INTEGRATION THEORY

In this chapter we give a self-contained presentation of the basic concepts of the theory of measure and integration. The principles discussed here and in Chapter 2 will serve as background for the study of probability as well as harmonic analysis, linear space theory, and other areas of mathematics.

1.1 INTRODUCTION

It will be convenient to start with a little practice in the algebra of sets. This will serve as a refresher and also as a way of collecting a few results that will often be useful.

Let $A_1, A_2, \ldots$ be subsets of a set Ω. If $A_1 \subset A_2 \subset \cdots$ and $\bigcup_{n=1}^{\infty} A_n = A$, we say that the A_n form an *increasing* sequence of sets with limit A, or that the A_n increase to A; we write $A_n \uparrow A$. If $A_1 \supset A_2 \supset \cdots$ and $\bigcap_{n=1}^{\infty} A_n = A$, we say that the A_n form a *decreasing* sequence of sets with limit A, or that the A_n decrease to A; we write $A_n \downarrow A$.

The *De Morgan laws*, namely, $\left(\bigcup_n A_n\right)^c = \bigcap_n A_n^c$, $\left(\bigcap_n A_n\right)^c = \bigcup_n A_n^c$, imply that

(1) if $A_n \uparrow A$, then $A_n^c \downarrow A^c$; if $A_n \downarrow A$, then $A_n^c \uparrow A^c$.

It is sometimes useful to write a union of sets as a disjoint union. This may be done as follows:

Let $A_1, A_2, \ldots$ be subsets of Ω. For each n we have

(2) $$\bigcup_{i=1}^{n} A_i = A_1 \cup (A_1^c \cap A_2) \cup (A_1^c \cap A_2^c \cap A_3) \cup \cdots \cup (A_1^c \cap \cdots A_{n-1}^c \cap A_n).$$

Furthermore,

(3) $$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (A_1^c \cap \cdots \cap A_{n-1}^c \cap A_n).$$

In (2) and (3), the sets on the right are disjoint. If the A_n form an increasing sequence, the formulas become

(4) $\bigcup_{i=1}^{n} A_i = A_1 \cup (A_2 - A_1) \cup \cdots \cup (A_n - A_{n-1})$

and

(5) $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (A_n - A_{n-1})$

(take A_0 as the empty set).

The results (1)–(5) are proved using only the definitions of union, intersection, and complementation; see Problem 1.

The following set operation will be of particular interest. If $A_1, A_2, \ldots$ are subsets of Ω, we define

(6) $\limsup_n A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$.

Thus $\omega \in \limsup_n A_n$ iff for every n, $\omega \in A_k$ for some $k \geq n$, in other words,

(7) $\omega \in \limsup_n A_n$ iff $\omega \in A_n$ for infinitely many n.

Also define

(8) $\liminf_n A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$.

Thus $\omega \in \liminf_n A_n$ iff for some n, $\omega \in A_k$ for all $k \geq n$, in other words,

(9) $\omega \in \liminf_n A_n$ iff $\omega \in A_n$ eventually, that is, for all but finitely many n.

We shall call $\limsup_n A_n$ the *upper limit* of the sequence of sets A_n, and $\liminf_n A_n$ the *lower limit*. The terminology is, of course, suggested by the analogous concepts for sequences of real numbers

$$\limsup_n x_n = \inf_n \sup_{k \geq n} x_k,$$

$$\liminf_n x_n = \sup_n \inf_{k \geq n} x_k.$$

See Problem 4 for a further development of the analogy.

The following facts may be verified (Problem 5):

(10) $(\limsup_n A_n)^c = \liminf_n A_n^c$

(11) $(\liminf_n A_n)^c = \limsup_n A_n^c$

(12) $\liminf_n A_n \subset \limsup_n A_n$

(13) If $A_n \uparrow A$ or $A_n \downarrow A$, then $\liminf_n A_n = \limsup_n A_n = A$.

In general, if $\liminf_n A_n = \limsup_n A_n = A$, then A is said to be the *limit* of the sequence $A_1, A_2, \ldots$; we write $A = \lim_n A_n$.

Problems

1. Establish formulas (1)–(5).
2. Define sets of real numbers as follows. Let $A_n = (-1/n, 1]$ if n is odd, and $A_n = (-1, 1/n]$ if n is even. Find $\limsup_n A_n$ and $\liminf_n A_n$.
3. Let $\Omega = \mathbb{R}^2$, A_n the interior of the circle with center at $((-1)^n/n, 0)$ and radius 1. Find $\limsup_n A_n$ and $\liminf_n A_n$.

4. Let $\{x_n\}$ be a sequence of real numbers, and let $A_n = (-\infty, x_n)$. What is the connection between $\limsup_{n\to\infty} x_n$ and $\limsup_n A_n$ (similarly for $\liminf$)?
5. Establish formulas (10)–(13).
6. Let $A = (a, b)$ and $B = (c, d)$ be disjoint open intervals of $\mathbb{R}$, and let $C_n = A$ if n is odd, $C_n = B$ if n is even. Find $\limsup_n C_n$ and $\liminf_n C_n$.

1.2 Fields, σ-Fields, and Measures

Length, area, and volume, as well as probability, are instances of the measure concept that we are going to discuss. A measure is a *set function*, that is, an assignment of a number $\mu(A)$ to each set A in a certain class. Some structure must be imposed on the class of sets on which μ is defined, and probability considerations provide a good motivation for the type of structure required. If Ω is a set whose points correspond to the possible outcomes of a random experiment, certain subsets of Ω will be called "events" and assigned a probability. Intuitively, A is an event if the question "Does ω belong to A?" has a definite yes or no answer after the experiment is performed (and the outcome corresponds to the point $\omega \in \Omega$). Now if we can answer the question "Is $\omega \in A$?" we can certainly answer the question "Is $\omega \in A^c$?," and if, for each $i = 1, \ldots, n$, we can decide whether or not ω belongs to A_i, then we can determine whether or not ω belongs to $\bigcup_{i=1}^n A_i$ (and similarly for $\bigcap_{i=1}^n A_i$). Thus it is natural to require that the class of events be closed under complementation, finite union, and finite intersection; furthermore, as the answer to the question "Is $\omega \in \Omega$?" is always "yes," the entire space Ω should be an event. Closure under *countable* union and intersection is difficult to justify physically, and perhaps the most convincing reason for requiring it is that a richer mathematical theory is obtained. Specifically, we are able to assert that the limit of a sequence of events is an event; see 1.2.1.

1.2.1 Definitions. Let $\mathscr{F}$ be a collection of subsets of a set Ω. Then $\mathscr{F}$ is called a *field* (the term *algebra* is also used) iff $\Omega \in \mathscr{F}$ and $\mathscr{F}$ is closed under complementation and finite union, that is,

(a) $\Omega \in \mathscr{F}$.
(b) If $A \in \mathscr{F}$, then $A^c \in \mathscr{F}$.
(c) If $A_1, A_2, \ldots, A_n \in \mathscr{F}$, then $\bigcup_{i=1}^n A_i \in \mathscr{F}$.

It follows that $\mathscr{F}$ is closed under finite intersection. For if $A_1, \ldots, A_n \in \mathscr{F}$, then

$$\bigcap_{i=1}^n A_i = \left(\bigcup_{i=1}^n A_i^c\right)^c \in \mathscr{F}.$$

If (c) is replaced by closure under *countable* union, that is,