HZ BOOKS

PEARSON Prentice Hall

经 典 原 版 书 库

实分析

(英文版 · 第3版)

Real Analysis

THIRD EDITION

H.L. ROYDEN

(美) H. L. Royden 斯坦福大学 著

机械工业出版社
China Machine Press

经典原版书库

实分析

（英文版·第3版）

Real Analysis

(Third Edition)

（美）H. L. Royden 著
斯坦福大学

机械工业出版社
China Machine Press

本书版权登记号：图字：01-2004-0819

图书在版编目（CIP）数据

实分析（英文版·第3版）/（美）罗伊登（Royden, H. L.）著. –北京：机械工业出版社，2004.3

（经典原版书库）

书名原文：Real Analysis, Third Edition

ISBN 7-111-13912-7

Ⅰ.实… Ⅱ.罗… Ⅲ.实分析－英文 Ⅳ.O174.1

中国版本图书馆CIP数据核字（2004）第006570号

机械工业出版社（北京市西城区百万庄大街22号 邮政编码 100037）

责任编辑：迟振春

北京昌平奔腾印刷厂印刷·新华书店北京发行所发行

2004年3月第1版第1次印刷

787mm × 1092mm 1/16 · 28.75印张

印数：0 001-3 000册

定价：45.00元

凡购本书，如有倒页、脱页、缺页，由本社发行部调换

本社购书热线：（010）68326294

Preface to the Third Edition

In the twenty years since this work was last revised it has contributed to the education of several generations of students, despite flaws in its treatment of Baire measure and the omission of invariant measures. I am therefore pleased by the opportunity of presenting a new edition ameliorating these shortcomings. Because of the difficulty of recapturing a point of view from the time of original composition, when the material was new to me, I have confined significant revisions to the theory of locally compact spaces and the study of measures on topological spaces. Elsewhere the alterations consist largely of minor improvements and the addition of new problems.

Part One is almost unchanged, the most notable change being the treatment of the Minkowski and Hölder inequalities. This treatment seems to me more natural, and it immediately gives the reversed inequalities for $0 < p < 1$. There are also relatively few changes in Chapters 11 and 12, the basic chapters on measure and integration. The principal additions consist of a section on integral operators and a section on Hausdorff measure added to Chapter 12.

Part Two sees somewhat more reorganization and extension: The sections on compact metric spaces and the Ascoli theorem have been moved from the chapter on compact spaces to the chapter on metric spaces, making these topics independent of the general theory of topological spaces. The material on Baire Category has been expanded with an indication of the principles used in applying this theory to proofs.

Chapter 8 on topological spaces is virtually unchanged, but Chapter 9 has been largely rewritten to expand the treatment accorded to locally compact spaces. Properties of locally compact spaces needed for measure theory are developed, and the concepts of paracompactness, exhaustion, and σ-compactness are discussed at length in the context of locally compact Hausdorff spaces. There is also a section on manifolds and the significance of paracompactness for them.

Chapter 10 is again little changed, with the exception of some material on convexity.

The material on Baire and Borel measures in locally compact spaces has been entirely rewritten. The treatment in the 2nd edition was seriously flawed. I do not think there were any actual misstatements of fact in the theorems and propositions, but the text was misleading, and a number of the statements in the problems were false. The difficulties, as in some other published treatments of measures in spaces that are not σ-compact, arose from problems of regularity. They were caused in my case by a misguided attempt to avoid talking about regularity directly. The current treatment meets these problems face-to-face and shows that one can have Baire (or Borel) measures that are inner regular or that are quasi regular but not always ones that are both. Included with this material is a direct proof of the Riesz–Markoff Theorem on the structure of positive linear functionals on $C_0(X)$. This proof is independent of the Daniell integral, allowing the chapter on the Daniell integral to be relegated to the end of the book.

Chapter 15 on automorphisms of measure spaces has been largely rewritten so that it now gives an extended treatment of Borel measures on complete separable metric spaces. I have tried to please my friend George Mackey by stressing the equivalence of these spaces with certain standard measure spaces, essentially Lebesgue measure on an interval of **R**.

The present edition contains a new chapter on invariant measures in Part Three. This topic was omitted from earlier editions because I was unsatisfied with the usual development of the theory. I thought the standard presentations of Haar measure awkward in the manner of their use of the Axiom of Choice to assure additivity, and I wanted to use instead a suitable generalization of the notion of limit along the lines used by Banach in the separable metric case. I also believed the proper context for invariant measures to be that of a transitive group of homeomorphisms on a locally compact space X. Thus the topology should be on the homeogeneous space X, with the group of homeomorphisms an abstract group without topology.

Of course, the group must satisfy some conditions in order that there should be a Baire measure on X invariant under the group. I introduce a property, called topological equicontinuity, and show that it suffices for the existence of an invariant measure. The unicity of such measures is considered in a number of particular cases, including that of locally compact topological groups. We also consider groups of diffeomorphisms and introduce the Hurwitz invariant integral when it exists. This integral has the advantage that one can give specific formulas for the integrand in many cases.

When this book was originally planned and written, the theory of Lebesgue integration was generally considered to be graduate level material, and the book was designed to be covered in a year-long course for first-year graduate students. Since that time the undergraduate curriculum has tended to include material on Lebesgue integration for advanced students, and this book has found increasing use at this level. The material presented here is of varying difficulty and sophistication. I have tried to arrange the chapters with considerable independence so the book will be useful for a variety of courses. One possibility for a short course is to cover Part One and Chapters 11 and 12. This gives a thorough treatment of integration and differentiation on **R** together with the fundamentals of abstract measure and integration. This could be supplemented by Chapter 7, covering metric spaces, and some topics on Banach spaces from Chapter 10. For students who are already familiar with basic measure and integration theory as well as the elements of metric spaces, one could construct a short course on measure and integration in topological spaces covering Chapters 9, 13, 14, and 15 with Chapters 7 and 8 as background material.

Were I writing the chapter on set theory today, I would give it a different tone, emphasize the various philosophical points of view about the foundations of mathematics and warn against endowing sets with reality and significance apart from the formal system in which they are embedded. The temptation to rewrite the chapter along these lines has been resisted, but I hope the readers of this book will ultimately read some of the many books on the foundations of mathematics before coming to a fixed opinion on the nature of infinite sets.

I wish to thank all of the diligent readers who have given me corrections and improvements over the last twenty years. My special thanks go to Jay Jorgenson and Hala Khuri for proofreading and checking the work in galley proof and to Elizabeth Arrington and

Elizabeth Harvey for turning large amounts of handwritten corrections and material into copy suitable for the printer.

H. L. R.

Stanford, California
July 1987

Preface to the Second Edition

This book is the outgrowth of a course at Stanford entitled "Theory of Functions of a Real Variable," which I have given from time to time during the last ten years. This course was designed for first-year graduate students in mathematics and statistics. It presupposes a general background in undergraduate mathematics and specific acquaintance with the material in an undergraduate course on the fundamental concepts of analysis. I have attempted to cover the basic material that every graduate student should know in the classical theory of functions of a real variable and in measure and integration theory, as well as some of the more important and elementary topics in general topology and normed linear space theory. The treatment of material given here is quite standard in graduate courses of this sort, although Lebesgue measure and Lebesgue integration are treated in this book before the general theory of measure and integration. I have found this a happy pedagogical practice, since the student first becomes familiar with an important concrete case and then sees that much of what he has learned can be applied in very general situations.

There is considerable independence among chapters, and the chart on page 4 gives the essential dependencies. The instructor thus has considerable freedom in arranging the material here into a course according to his taste. Sections that are peripheral to the principal line of argument have been starred (*). The Prologue to the Student lists some of the notations and conventions and makes some suggestions.

The material in this book belongs to the common culture of mathematics and reflects the craftsmanship of many mathematicians. My treatment of it is particularly indebted to the published works of Constantine Carathéodory, Paul Halmos, and Stanislaw Saks and to the lectures and conversations of Andrew Gleason, John Herriot, and Lynn Loomis. Chapter 15 is the result of intensive discussion with John Lamperti.

I also wish to acknowledge my indebtedness for helpful suggestions and criticism from numerous students and colleagues. Of the former I should like to mention in particular Peter Loeb, who read the manuscript of the original edition and whose helpful suggestions improved the clarity of a number of arguments, and Charles Stanton, who read the manuscript for this revision, correcting a number of fallacious statements and problems. Among my colleagues, particular thanks are due to Paul Berg, who pointed out Littlewood's "three principles" to me, to Herman Rubin, who provided counterexamples to many of the theorems the first time I taught the course, and to John Kelley, who read the manuscript, giving helpful advice and making me omit my polemical remarks. (A few have reappeared as footnotes, however.) Finally, my thanks go to Margaret Cline for her patience and skill in transforming illegible copy into a finished typescript for the original edition, to William Glassmire for reading proofs of the revised edition, to Valerie Yuchartz for typing the material for this edition, and to the editors at Macmillan for their forbearance and encouragement during the years in which this book was written.

H. L. R.

Stanford, California
September 1967

Contents

Part Three
GENERAL MEASURE AND INTEGRATION THEORY

Prologue to the Student

This book covers a portion of the material that every graduate student in mathematics must know. For want of a better name the material here is denoted by real analysis, by which I mean those parts of modern mathematics which have their roots in the classical theory of functions of a real variable. These include the classical theory of functions of a real variable itself, measure and integration, point-set topology, and the theory of normed linear spaces. This book is accordingly divided into three parts. The first part contains the classical theory of functions, including the classical Banach spaces. The second is devoted to general topology and to the theory of general Banach spaces, and the third to abstract treatment of measure and integration.

Prerequisites. It is assumed that the reader already has some acquaintance with the principal theorems on continuous functions of a real variable and with Riemann integration. No formal use of this knowledge is made here, and Chapter 2 provides (formally) all of the basic theorems required. The material in Chapter 2 is, however, presented in a rather brief fashion and is intended for review and as introduction to the succeeding chapters. The reader to whom this material is not already familiar may find it difficult to follow the presentation here. We also presuppose some acquaintance with the elements of modern algebra as taught in the usual undergraduate course. The definitions and elementary properties of groups and rings are used in some of the peripheral sections, and the basic notations of linear vector spaces are used in Chapter 10. The theory of sets underlies all of the material in this book, and I have sketched some of the basic facts from set theory in Chapter 1. Since the remainder of the book is full of applications of set theory, the students should become adept at set-theoretic arguments in progressing through the book. I recommend that the student first read Chapter 1 lightly and then refer back to it as needed. The books by Halmos [6][1] and Suppes [14] contain a more thorough treatment of set theory and can be profitably read by the student while reading this book.

Logical notation. It is convenient to use some abbreviations for logical expressions. We use '&' to mean 'and' so that 'A & B' means 'A and B'; '$\vee$' means 'or' so that '$A \vee B$' means 'A or B (or both)'; $\neg$ means 'not' or 'it is not the case that', so that '$\neg A$' means 'it is not the case that A'. Another important notation is the one that we express by the symbol '$\Rightarrow$'. It has a number of synonyms in English, so that the statement '$A \Rightarrow B$' can be expressed by saying 'if A, then B', 'A implies B', 'A only if B', 'A is sufficient for B', or 'B is necessary for A'. The statement '$A \Rightarrow B$' is equivalent to each of the statements '$(\neg A) \vee B$' and '$\neg(A \;\&\; (\neg B))$'. We also use the notation '$A \Leftrightarrow B$' to mean '$(A \Rightarrow B) \;\&\; (B \Rightarrow A)$'. English synonyms for '$A \Leftrightarrow B$' are 'A if and only if B', 'A iff B', 'A is equivalent to B', and 'A is necessary and sufficient for B'.

In addition to the preceding symbols we use two further abbreviations: '(x)' to mean 'for all x' or 'for every x', and '$(\exists x)$' to mean 'there is an x' or 'for some x'. Thus the statement $(x)(\exists y)(x < y)$ says that for every x there is a y which is larger than x. Similarly, $(\exists y)(x)(x < y)$ says that there is a y which is larger than every x. Note that these two statements are different: As applied to real numbers, the first is true and the second is false.

Since saying that there is an x such that $A(x)$ means that it is not the case that for every x we have $\neg A(x)$, we see that $(\exists x)A(x) \Leftrightarrow \neg(x)\neg A(x)$. Similarly, $(x)A(x) \Leftrightarrow \neg(\exists x)\neg A(x)$. This rule is often convenient when we wish to express the negative of a complex statement. Thus

$$\begin{aligned}\neg\{(x)(\exists y)(x < y)\} &\Leftrightarrow \neg(x)\neg(y)\neg(x < y)\\ &\Leftrightarrow (\exists x)(y)\neg(x < y)\\ &\quad(\exists x)(y)(y \leq x),\end{aligned}$$

[1] Numbers in brackets refer to the Bibliography, p. 435.

where we have used properties of the real numbers to infer that $\neg(x < y) \Leftrightarrow (y \leq x)$.

We sometimes modify the standard logical notation slightly and write $(\epsilon > 0)\,(\ldots)$, $(\exists\delta > 0)\,(\ldots)$, and $(\exists x \in A)\,(\ldots)$ to mean 'for every ϵ greater than 0 (...)', 'there is a δ greater than 0 such that (...)', and 'there is an x in the set A such that (...)'. This modification shortens our expressions. For example, $(\epsilon > 0)\,(\ldots)$ would be written in standard notation $(\epsilon)\,\{(\epsilon > 0) \Rightarrow (\ldots)\}$.

For a thorough discussion of the formal use of logical symbolism, the student should refer to Suppes [14].

Statements and their proofs. Most of the principal statements (theorems, propositions, etc.) in mathematics have the standard form 'if A, then B' or in symbols '$A \Rightarrow B$'. The *contrapositive* of $A \Rightarrow B$ is the statement $(\neg B) \Rightarrow (\neg A)$. It is readily seen that a statement and its contrapositive are equivalent; that is, if one is true, then so is the other. The direct method of proving a theorem of the form '$A \Rightarrow B$' is to start with A, deduce various consequences from it, and end with B. It is sometimes easier to prove a theorem by contraposition, that is, by starting with $\neg B$ and deriving $\neg A$. A third method of proof is proof by contradiction or *reductio ad absurdum:* We begin with A and $\neg B$ and derive a contradiction. All students are enjoined in the strongest possible terms to eschew proofs by contradiction! There are two reasons for this prohibition: First, such proofs are very often fallacious, the contradiction on the final page arising from an erroneous deduction on an earlier page, rather than from the incompatibility of A with $\neg B$. Second, even when correct, such a proof gives little insight into the connection between A and B, whereas both the direct proof and the proof by contraposition construct a chain of argument connecting A with B. One reason that mistakes are so much more likely in proofs by contradiction than in direct proofs or proofs by contraposition is that in a direct proof (assuming the hypothesis is not always false) all deductions from the hypothesis are true in those cases where the hypothesis holds, and similarly for proofs by contraposition (if the conclusion is not always true) the deductions from the negation of the conclusion are true in those cases where the conclusion is false. Either way, one is dealing with true statements, and one's intuition and knowledge about what is true help to keep one from making erroneous statements. In proofs by contradiction, however, you are (assuming the theorem true) in the unreal world where any statement can be derived, and so the falsity of a statement is no indication of an erroneous deduction.