Hemen Dutta · Billy E. Rhoades *Editors*

Current Topics in Summability Theory and Applications

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Preface

This book is intended for graduate students and researchers who have interest in functional analysis, in general and summability theory, in particular. It describes several useful topics in summability theory along with applications. The book consists of nine chapters and is organized as follows:

Chapter "An Introduction to Summability Methods" is introductory in nature. This chapter focuses on the historical development of summability theory right from Cauchy's concept to till date. Summability methods developed from the two basic processes—T-process and ϕ-process—have also been discussed in this chapter.

Chapter "Some Topics in Summability Theory" deals with the study of some classical and modern summability methods, and the connections among them. In fact, results concerning summability by weighted mean method, the (M, λ_n) method, the Abel method, and the Euler method are presented. Then the sequence space Λ_r, $r \geq 1$ being a fixed integer, is defined and a Steinhaus type theorem is proved. The space Λ_r is then studied in the context of sequences of 0's and 1's. Further, the core of a sequence is studied, an improvement of a result of Sherbakhoff is proved and a very simple proof of Knopp's core theorem is then deduced. Finally, a study of the matrix class (ℓ, ℓ) is presented.

Chapter "Summability and Convergence Using Ideals" is concentrated on different concepts of summability and convergence using the notions of ideals and essentially presents the basic developments of these notions. This chapter starts with the first notion of ideal convergence and goes on to discuss in detail how the notion has been extended over the years from single sequences to double sequences and nets. This chapter also discusses some of the most recent advances made in this area, in particular applications of ideal convergence to the theory of convergence of sequences of functions. Some problems are also listed which still remain open.

In chapter "Convergence Acceleration and Improvement by Regular Matrices", a new, non-classical convergence acceleration concept is studied and compared with the well-known classical convergence acceleration concept. It is shown that the new concept allows to compare the speeds of convergence for a larger set of sequences than the classical convergence acceleration concept. Also, regular matrix methods that improve and accelerate the convergence of sequences and series are studied. The results described in this chapter are further applied to increase the order of approximation of Fourier expansions and Zygmund means of Fourier expansions in certain Banach spaces.

Chapter "On Summability, Multipliability and Integrability" deals with the study of summability and multipliability of vector families indexed by well-ordered sets of real numbers. These concepts generalize the classical notions of convergence of infinite series and products. The studies are also motivated by problems in integration theory of functions of one variable. In particular, the chapter describes the relation between integrability and product integrability on the one side, and summability and multipliability on the other side. Applications in the theory of differential equations with impulses and distributional differential equations are presented, and concrete examples are introduced to illustrate the derived theoretical results.

In chapter "Multi-dimensional Summability Theory and Continuous Wavelet Transform", the connection between multi-dimensional summability theory and continuous wavelet transform is investigated. Two types of θ-summability of Fourier transforms are considered, the circular and rectangular summability. Norm and almost everywhere convergence of the θ-means are shown for both types. The inverse wavelet transform is traced back to summability means of Fourier transforms. Using the results concerning the summability of Fourier transforms, norm and almost everywhere convergence of the inversion formula are obtained for functions from the L_p and Wiener amalgam spaces.

In chapter "Absolute Riesz and Related Summability Methods", several theorems dealing with the absolute Riesz summability of infinite series have been given. Additionally, some theorems which are generalization of these theorems to absolute matrix summability have been given by using several different sequences.

Chapter "Some Applications of Summability Theory" discusses some applications of summability theory in sequence spaces defined by certain functions and summability methods, which are related to statistical convergence and their applications. Several topological and geometric properties of the sequence spaces, such as the (β)-property, Banach–Saks property, Kadec–Klee property, Opial property, etc., are also discussed. Then some applications of summability theory to Tauberian theorems, both in ordinary sense and in statistical sense are discussed. Finally, some results related to the Tauberian theory characterized by weighted summability methods such as, the generalized de la Vallée–Poussin method, generalized Nörlund-Cesàro, etc., are presented.

Chapter "Degree of Approximation of Functions Through Summability Methods" first discusses a result on the degree of approximation of functions belonging to the $Lip(\alpha, r)$ class, using almost Riesz summability method of its infinite Fourier series. Then a result concerning the degree of approximation of the conjugate of a function f belonging to $Lip(\xi(t), r)$ class by Euler (E, q) summability of conjugate series of its Fourier series has been established. The results discussed in this chapter generalize several existing results.

The editors are very much thankful to all learned referees for their valuable and helpful suggestions, and friends for encouragement and moral support.

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Contents

An Introduction to Summability Methods

U.K. Misra

Abstract This chapter contains four sections. First section is introductory in which a brief description of the development of the subject is presented. In the second section, the basic technique of the summability method has been discussed. As the summability methods are considered to be derived from two general processes, in section three the two summability processes and their characterizations have been presented. Section four is devoted to different methods of summabilties which are derived from the two basic processes and their properties have been discussed. The summability methods such as matrix summability, Cesàro summability, Hölder summability, Harmonic summability, Generalized Cesàro summability, Riesz's typical means summability, Nörlund summability, Riesz's summability, generalized Nörlund summability, indexed summability, Abel summability, Euler summability, Borel summability, Hausdorff summability, and Banach summability methods have been discussed in sequel.

Keywords Infinite series · Sequence · Summability methods · Absolute summability · Indexed summability

1 Introduction

The concept of "the infinity" seems to have excited human thought right from the time, man tried to put his intellect to his observations. The consciousness of the limitless expanse of space, the awareness of the everlasting stream of time and the experience of non-terminating chain of counting numbers are some of the fascinating aspects of realities which astounded the human mind. They did direct human imagination towards the concept of unattainable and limitless infinity.

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With the introduction of algebraic operations in the domain of number system, there emerged the concept of "infinite series". The uses of such series can be traced far back into the realm of the history of mathematics itself. These were inherent in the methods of exhaustion restored to by Greek mathematicians for finding lengths of curves, areas bounded by the simple curves and volumes of simple solid bodies. However, the precise concept of the sum of an infinite series remained obscure until the recent past. In the beginning, while using infinite series, there was a tendency to interpret the concepts of an infinite sum as an extension of a finite sum. Indeed, in the absence of a clear concept of infinite series, mathematicians tended to believe that all the rules applicable to a finite sum would as such be applicable to infinite series too. But then applications of such concepts to infinite series led many times to irreconcilable situations for which there had been no satisfactory explanations. It may be of interest to note that, while some times arithmetic operations when applied to certain infinite series worked very well, yet the same applied to some other series led to paradoxical situations like $1 = 0$.

In the seventeenth century, James Gregory worked on infinite series and published several papers on Maclaurin series. He termed the former class of infinite series as "convergent series" [8]. The aim of such a classification was to caution mathematicians against the uses of "non-convergent series," which may bring in a contradiction. The mathematicians prior to the time of Leonard Euler (1707–1783), used only convergent series and carefully avoided the use of non-convergent ones. And the mystery of the anomalous behavior of infinite series remained obscure for quite sometime.

In the eighteenth century, Leonhard Euler (1707–1783) developed the theory of hyper-geometric series and q-series and gave more of an idea about infinite and non-convergent series. Mathematicians prior to him had used only convergent series and carefully avoided the use of non-convergent ones, and the mystery of the anomalous behavior of infinite series remained obscure for quite some time.

Carl Friedrich Gauss, the German Mathematician, was a pioneer in the introduction of the concept of infinite into Mathematical Analysis. However, the credit for clearly defining the sum of an infinite series goes to the French mathematician A.L. Cauchy (1789–1857), who had been a pioneer in introducing rigor into mathematical analysis. It was he who crystallized the concept of limit in definite terms. In 1821, Cauchy formalized ideas concerning convergence and divergence of infinite series. He clearly defined the sum of an infinite series based on the concept of limit developed by him in his book entitled "Analyse Algèbrique". This sum is known as the "natural sum" or Cauchy's sum of a series.

Let $\{u_n\}$, be a given real- or complex-valued sequence. Then an expression of the form

$$u_1 + u_2 + u_3 + u_4 + \cdots \tag{1}$$

is called an "infinite series" and is generally denoted by

$$\sum_{n=1}^{\infty} u_n \text{ or } \sum u_n \tag{2}$$

in brief. If all of the terms of the sequence $\{u_n\}$ after a certain number are zero, then the expression

$$u_1 + u_2 + u_3 + u_4 + \cdots + u_m \tag{3}$$

is called a "finite series" and is written simply as

$$\sum_{n=1}^{m} u_n. \tag{4}$$

An expression of the form

$$\sum_{n=1}^{\infty} u_n = \sum u_n = u_1 + u_2 + u_3 + u_4 + \cdots, \tag{5}$$

which involves the addition of infinitely many terms, has indeed no meaning, as there is no way to sum an infinite number of terms. However, in order to accord some plausible meaning to such an expression, Cauchy uses the concept of "limits". For this Cauchy forms a sequence of partial sums of the series and defines the sum

$$u_1 + u_2 + u_3 + u_4 + \cdots \tag{6}$$

as the limiting value of the partial sums as the number of terms tend to infinity.

Let $\sum u_n$ be an infinite series with real or complex terms and let, for $n = 1, 2, 3, \ldots$,

$$s_n = u_1 + u_2 + u_3 + u_4 + \cdots + u_n \tag{7}$$

Then s_n is called the nth partial sum of the series and the sequence $\{s_n\}$, thus obtained, is called the sequence of partial sums of the series $\sum u_n$. An infinite series $\sum u_n$ is said to converge, diverge or oscillate, according as its sequence of partial sums $\{s_n\}$ converges, diverges or oscillates. According to Cauchy the infinite series $\sum u_n$ has the sum 's' (known as a Cauchy sum) if and only if there exists a finite real number 's' such that, for every $\epsilon > 0$, there exists a natural number n_0 such that

$$|s_n - s| < \epsilon, \quad \text{for every} \quad n \geq n_0. \tag{8}$$

That is to say, $\lim_{n\to\infty} s_n = s$. A series for which Cauchy's sum exists (that is, $\lim_{n\to\infty} s_n = s$, a finite number) is termed as convergent. It was easily verified that series classified as convergent by Gregory were all convergent in the sense of Cauchy also.

The series which are not convergent, that is, the series having no sum in the sense of Cauchy, were termed as "divergent." According to Cauchy divergent series do not belong to the understandable domain of mathematics and the convergent series were the only valid mathematical entities. Before Cauchy, series, convergent, and divergent were both in use and no distinction was made between the two. This led to paradoxes and irreconcilable situations. But Cauchy, in one stroke, removed all of the contradictions and paradoxes, by outcasting divergent series from the valid domain of mathematics. It brought much needed relief to the-then mathematicians, whose faith in their methodology was badly shaken, owing to the frequent appearances of paradoxes and contradictions. After this, it began to be regarded that the problem of the sum of an infinite series had fully and finally been resolved. Thus, even though divergent series were used to good purposes earlier by such eminent mathematicians as Leibnitz, Euler and others, yet they were thrown out from the valid domain of mathematics without hesitation. The concept of the sum of an infinite series, as defined by Cauchy, was so natural, so efficacious that mathematicians thought that the problem of the sum of the infinite series had finally been settled once and for all.

Abel (a Norwegian Mathematician, 1802–1829) [1] was another important contributor for giving ideas concerning convergence and divergence in the early part of nineteenth century. He was so excited with the discovery, that, in a letter to Holmbee, expressed his conviction in such telling words as "Les series divergents sont, en general, quelque choose de bien fatal et c est une honte qu on ose y fonder aucune demonstration" (Divergent series are, in general, sometimes quite calamitous and it is a shame that any one dares to base a proof on them.).

Since mathematics is based on principles of reasoning, any slightest deviation from the right track of the flow of mathematical ideas would ultimately end in disharmony. Even after the theory propounded by Cauchy had received the stamp of finality of almost all of the mathematicians of the time, it did face the same disharmonies particular to the field of orthogonal expansion of continuous functions and product series. It was noted that certain non-convergent series (Fourier series) behaved very much in the same way with regard to arithmetical operations on them as convergent ones, and the calculation based on certain asymptotic series, not convergent in the sense of Cauchy, used in dynamical astronomy, were quite valid and verifiable otherwise. All these facts, in course of time led mathematicians to conclude that the Cauchy method of assigning a sum to an infinite series was of far-reaching importance, and yet was not that devilish as they were earlier made out to be. All these situations stirred the imagination of several inquisitive mathematicians to develop, into the character of the sum of an infinite series, over and above that of Cauchy, of assigning sum. Persistent efforts made by a number of eminent mathematicians led to the discovery of alternative methods, which were closely connected to that of Cauchy, yet associated sums, even to divergent series, particularly to those whose partial sums oscillate. By the close of the nineteenth century, several alternative methods of assigning sums to infinite series were invented by mathematicians. These methods of summation were termed "Summability Methods."

Some of the most familiar methods of summability are those that are associated with the names of great mathematicians like Abel, Borel, Cesàro, Euler, Hausdorff, Hölder, Lambert, Nörlund, Reisz, Riemann, and Lebesgue. Thus, by the third decade of the twentieth century, a very rich and fruitful theory of summability had been introduced. This theory found applications even in such remote fields as probability and the theory of numbers. Norbert Wiener applied Lambert's method of summation to prove the prime number theorem.

As Cauchy's concept of sum of a convergent series well withstood all of the rigors of mathematics, the framework of the summability methods was, in general, so devised as to assign convergent series, the same sum as that assigned by Cauchy. This leads to the following terminologies.

Definition 1.1 A summability method is a function from the set of sequences of partial sums of a series to a value. Thus, in its broadest meaning, summability is the theory of the assigning of limits which is fundamental in analysis, function theory, topology, and functional analysis.

Definition 1.2 A summability method is said to be regular if the method sums all convergent series to its Cauchy's sum [9].

Definition 1.3 Two summability methods are said to be consistent if they assign the same sum to the same series [9].

Thus, regular methods of summability may be regarded as a generalization of Cauchy's concept of convergence. Just as the concept of ordinary convergence has been generalized into that of summability, commonly termed as "Ordinary Summability," the concept of absolute convergence too has been extended similarly into a concept called as "Absolute Summability."

2 Basic Technique

The basic technique of all summability methods is to transform a given infinite series or sequence of partial sums, into another series, or sequence, on which Cauchy's method is applicable. The transformation chosen is usually linear and is such that it preserves Cauchy's sum when applied to convergent series. Further, a transformation, to be worthwhile, should be such as to transform some divergent series too into ones on which Cauchy's method of assigning sum cannot be applicable.

Thus, if T is a transformation which represents a summability method, then it should have the following properties:

(i) If $\sum a_n$ is a convergent series with sum 's', then $T \sum a_n$ is also convergent having the same sum 's'.
(ii) If $\sum a_n$ and $\sum b_n$ are two series and p and q are real or complex numbers, then

$$T(p \sum a_n + q \sum b_n) = p(T \sum a_n) + q(T \sum b_n). \tag{9}$$

(iii) The T-method is able to assign a sum to at least one infinite series for which Cauchy's method fails.

Conditions (i), (ii), and (iii) are called the regularity conditions, the linear condition, and range condition, respectively.

3 Basic Process

All summability methods are considered to be derived from the following two general basic processes:

(i) Methods based on a sequence-to-sequence transformation, usually termed as a T-process and
(ii) Methods based on a sequence-to-function transformation, usually termed as a ϕ-process.

3.1 T-Process

Summability methods, in which the sequence of partial sums of an infinite series is transformed into another sequence, constitute a T-process. They are usually called sequence-to-sequence transformation methods.

Let $\sum u_n$ be an infinite series with sequence of partial sums $\{s_n\}$. Then $\sum u_n$ has the Cauchy's sum if $\lim_{n\to\infty} s_n = s$, where '$s$' is a finite number. Let T be a linear transformation and let $\{t_n\} = \{T(s_n)\}$. Then the T-method consists in the formulation of an auxiliary sequence $\{t_n\}$, obtained by a sequence-to-sequence transformation.

In analogy with Cauchy's method, we say that a series $\sum u_n$ is summable by a T-method to the sum 's' if and only if

$$\lim_{n\to\infty} t_n = s. \tag{10}$$

Further, we say that the series $\sum u_n$ is absolutely convergent if $\sum |u_n| < \infty$, which is same as

$$\sum |s_n - s_{n-1}| < \infty. \tag{11}$$

That is, the sequence $\{s_n\}$ is of bounded variation. Following the same analogy, the series $\sum u_n$ is said to be absolutely summable by a T-method or simply $|T|$-summable, if and only if the auxiliary sequence $\{t_n\}$ is of bounded variation: that is

$$\sum |t_n - t_{n-1}| < \infty. \tag{12}$$