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Trends in Differential Equations and Applications

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Preface

During recent years applied mathematics techniques have attained considerable dissemination within the experimental sciences and engineering. Special attention has been devoted to biomathematics and medicine, including the analysis of mathematical models for the description of tumors, blood flux in arteries, the heart and flow patterns inside an aneurysm dome. A significant element of this dissemination also derives from the applications of mathematics in industry. International meetings such as those of the European Consortium for Mathematics in Industry (ECMI) and the International Council for Industrial and Applied Mathematics (ICIAM) bear witness to these advances. In Spain, various research groups have contributed to this development; most have been based in universities across the country, sometimes acting in collaboration with nonpublic laboratories. Links and coordination with foreign groups and universities have also proved essential. The significance of the Spanish contribution is reflected in the fact that the next ECMI meeting will take place in Santiago de Compostela in June 2016, while Valencia will host the next ICIAM congress in 2019.

The XXIVth Congress on Differential Equations and Applications/XIVth Congress on Applied Mathematics was held in Cádiz (a city founded more than three millennia ago), Spain, from 8 to 12 June 2015. This biennial international conference is the most important event organized by the Spanish Society of Applied Mathematics (SEMA). Any information on the conference is available on the Society website: http://www.sema.org.es/web/index.php. The conference brought together an excellent group of international and national researchers interested in the different branches of applied mathematics. Topics ranged from tsunami prediction to modeling of epidemiological processes and encompassed mathematics in architecture, high-order long-term integration of dynamical systems, the search for exact solutions of ordinary differential equations, oceanography, numerical acoustics, mathematics in industry, numerical linear algebra, and so on. This wide variety of subject matter reflects the multidisciplinary nature of the various research projects being carried out at present by both Spanish teams and groups in other countries

The collection of articles in this book represents a selection of the contributions presented at this conference in Cádiz. Every submitted paper has undergone a standard refereeing process. The volume provides a good summary of the recent activity of the various Spanish research groups interested in the applications of mathematics to different branches of the experimental sciences and engineering.

The publication has been made possible by the contributions of a number of people. First of all, we would like to thank the authors themselves for submitting their work. Special thanks are due to the referees who agreed to participate: their comments and suggestions have resulted in improvements in most of the included contributions. Finally, we would like to express our gratitude to Francesca Bonadei from Springer for the patience, attention and support that she has shown at every stage of the editorial process.

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Contents

Approximate Osher-Solomon Schemes for Hyperbolic Systems

M.J. Castro, J.M. Gallardo, and A. Marquina

Abstract The Osher-Solomon scheme is a classical Riemann solver which enjoys a number of interesting features: it is nonlinear, complete, robust, entropy-satisfying, smooth, etc. However, its practical implementation is rather cumbersome, computationally expensive, and applicable only to certain systems (compressible Euler equations for ideal gases or shallow water equations, for example). In this work, a new class of approximate Osher-Solomon schemes for the numerical approximation of general conservative and nonconservative hyperbolic systems is proposed. They are based on viscosity matrices obtained by polynomial or rational approximations to the Jacobian of the flux evaluated at some average states, and only require a bound on the maximal characteristic speeds. These methods are easy to implement and applicable to general hyperbolic systems, while at the same time they maintain the good properties of the original Osher-Solomon solver. The numerical tests indicate that the schemes are robust, running stable and accurate with a satisfactory time step restriction, and the computational cost is very advantageous with respect to schemes using a complete spectral decomposition of the Jacobians.

1 Introduction

The Osher-Solomon scheme, introduced in [12], is a nonlinear and complete Riemann solver enjoying a number of interesting features: it is robust, entropy-satisfying, smooth, and has a good behavior with slowly-moving shocks. Its main drawback is that it requires the computation of a path-dependent integral in phase space, leading to a very complex and computationally expensive Riemann solver.

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Due to this difficulties, its practical application has been restricted to certain systems, e.g., the compressible Euler equations [15].

In [7], Dumbser and Toro introduced a reformulated version of the Osher-Solomon solver, denoted as DOT (Dumbser-Osher-Toro), in which the integrals in phase space are numerically approximated by means of a Gauss-Legendre quadrature formula. This leads to a scheme much simpler than the original one and applicable to general hyperbolic systems. In particular, the viscosity matrix of the numerical flux is defined as a linear combination of the absolute value matrix of the physical flux evaluated at certain quadrature points. The computation of these absolute value matrices requires the knowledge of the complete eigenstructure of the system. Thus, the scheme may be computationally expensive for systems in which the eigenstructure is not known or difficult to compute.

In this work we propose an alternative version of the DOT solver, in which the absolute value matrices are approximated using appropriate functional evaluations of the Jacobian of the flux evaluated at the quadrature points. These schemes only require a bound on the maximum speed of propagation, thus avoiding the computation of the full eigenstructure of the system. Several families of approximations have been considered. The first one is based on Chebyshev polynomials, which provide optimal uniform approximations to the absolute value function. On the other hand, it is well-known that rational functions provide more precise approximations to $|x|$ than polynomial functions. For this reason, two different families of rational approximations have also been used, based on Newman [10] and Halley [4] functions. This families of functions have also been considered in the recently introduced RVM schemes (see [6]).

The proposed approximate Osher-Solomon schemes have been applied to a number of initial value Riemann problems for ideal magnetohydrodynamics, to observe their behavior with respect to some challenging scenarios in numerical simulations. The numerical tests indicate that our schemes are robust, stable and accurate with a satisfactory time step restriction. Comparisons with the DOT solver and some other well-known schemes in the literature (e.g., Roe and HLL) have also been performed.

2 Preliminaries

Consider a hyperbolic system of conservation laws

$$\partial_t w + \partial_x F(w) = 0, \tag{1}$$

where $w(x,t)$ takes values on an open convex set $\Omega \subset \mathbb{R}^N$ and $F: \Omega \to \mathbb{R}^N$ is a smooth flux function. We are interested in the numerical solution of the Cauchy

problem for (1) by means of finite volume methods of the form

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2} - F_{i-1/2}), \tag{2}$$

where w_i^n denotes the approximation to the average of the exact solution at the cell $I_i = [x_{i-1/2}, x_{i+1/2}]$ at time $t^n = n\Delta t$ (the dependence on time will be dropped unless necessary). We assume that the numerical flux is given by

$$F_{i+1/2} = \frac{F_i + F_{i+1}}{2} - \frac{1}{2}Q_{i+1/2}(w_{i+1} - w_i), \tag{3}$$

where $F_i = F(w_i)$ and $Q_{i+1/2}$ denotes the numerical *viscosity matrix*, which determines the numerical diffusion of the scheme.

The condition of hyperbolicity of system (1) states that the Jacobian matrix of the flux at each state $w \in \Omega$,

$$A(w) = \frac{\partial F}{\partial w}(w),$$

can be diagonalized as $A = PDP^{-1}$, where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$, λ_i being the eigenvalues of A, and the matrix P is composed by the associated right eigenvalues of A. As it is usual, we denote the positive and negative parts of A, respectively, as $A^+ = PD^+P^{-1}$ and $A^- = PD^-P^{-1}$, where $D^\pm = \mathrm{diag}(\lambda_1^\pm, \ldots, \lambda_N^\pm)$, with $\lambda_i^+ = \max(\lambda_i, 0)$ and $\lambda_i^- = \min(\lambda_i, 0)$. It is clear that $A = A^+ + A^-$. On the other hand, the absolute value of A is defined as $|A| = A^+ - A^-$.

It is interesting to note that the well-known Roe's method [13] can be written in the form (3) with viscosity matrix $Q_{i+1/2} = |A_{i+1/2}|$, where $A_{i+1/2}$ is a Roe matrix for the system. Several numerical methods have been developed by using approximations to $|A_{i+1/2}|$ as viscosity matrices. A general approach to build such kind of approximations by means of polynomial and rational functions has recently been introduced in [5, 6]. In particular, it has been shown that a number of well-known schemes in the literature can be viewed as particular cases within this general framework: Roe, Lax-Friedrichs, Rusanov, HLL, FORCE, and many others.

3 The Osher-Solomon Scheme

The Osher-Solomon scheme [12] is a nonlinear Riemann solver that possesses a number of interesting features: it is entropy-satisfying, robust, differentiable and good behaved for slowly-moving shocks. On the contrary, its implementation is rather cumbersome, computationally expensive, and only applicable to certain systems.

Let $A(w)$ be the Jacobian of F evaluated at w, and assume the flux splitting

$$F(w) = F^+(w) + F^-(w), \tag{4}$$

where

$$A^{\pm}(w) = \frac{\partial F^{\pm}}{\partial w}(w).$$

The *classical Osher-Solomon numerical flux* is then defined as

$$F_{i+1/2} = F^+(w_i) + F^-(w_{i+1}).$$

Let now Φ be a path in the phase-space Ω linking the states w_i and w_{i+1}, i.e., $\Phi: [0, 1] \to \Omega$ is a Lipschitz continuous function such that $\Phi(0) = w_i$ and $\Phi(1) = w_{i+1}$. Then, we can write

$$F^-(w_{i+1}) - F^-(w_i) = \int_0^1 A^-(\Phi(s))\Phi'(s)ds,$$

from which we deduce

$$F_{i+1/2} = F_i + \int_0^1 A^-(\Phi(s))\Phi'(s)ds. \tag{5}$$

Similarly, we could also write

$$F_{i+1/2} = F_{i+1} - \int_0^1 A^+(\Phi(s))\Phi'(s)ds. \tag{6}$$

Combining (5) and (6), the Osher-Solomon flux can be written as

$$F_{i+1/2} = \frac{F_i + F_{i+1}}{2} - \frac{1}{2}\int_0^1 |A(\Phi(s))|\Phi'(s)ds. \tag{7}$$

The expression (7) for the numerical flux depends on the path Φ in phase-space, so in general it may be difficult to compute. Osher and Solomon [12] proposed a way to build, under certain assumptions, a path which makes possible to perform the integration. Unfortunately, the resulting solver is rather complex, computationally expensive, and only applicable to certain systems.

In [7] the authors propose a way to circumvent the drawbacks of the Osher-Solomon solver, maintaining at the same time its good features. First, the path consisting in segments is chosen:

$$\Phi(s) = w_i + s(w_{i+1} - w_i), \quad s \in [0, 1].$$

Thus (7) can be written in the form (3), with viscosity matrix

$$Q_{i+1/2} = \int_0^1 \left|A(w_i + s(w_{i+1} - w_i))\right| ds.$$

To avoid the analytical integration, the integral is evaluated numerically using a Gauss-Legendre quadrature formula. The resulting numerical flux, denoted as DOT (Dumbser-Osher-Toro), has the form (3) with viscosity matrix given by

$$Q_{i+1/2} = \sum_{k=1}^{q} \omega_k \left|A(w_i + s_k(w_{i+1} - w_i))\right|, \tag{8}$$

where $s_k \in [0, 1]$ and ω_k are the weights of the quadrature formula. The resulting scheme is simple to implement and applicable to general hyperbolic systems. On the other hand, it needs the full eigenstructure of the system, which must be computed numerically when it is not known or difficult to calculate.

4 Approximate Osher-Solomon Schemes

With the aim of simplifying the computation of the DOT numerical viscosity matrix (8), it would be desiderable to approximate the intermediate matrices

$$\left|A(w_i + s_k(w_{i+1} - w_i))\right|, \quad k = 1, \ldots, q,$$

in a simple and efficient way. Two approaches will be considered in this section, one based on Chebyshev polynomials and another relying on rational approximations.

Let $P(x)$ be a polynomial approximation to the absolute value function $|x|$ in the interval $[-1, 1]$, satisfying the *stability condition* [5]

$$|x| \leq P(x) \leq 1, \quad \forall\, x \in [-1, 1]. \tag{9}$$

For a given matrix A, if $\lambda_{\max}$ is the eigenvalue of A with maximum absolute value (or an upper bound of it), $|A|$ can be approximated as

$$|A| \approx |\lambda_{\max}| P\big(|\lambda_{\max}|^{-1} A\big).$$

Denote

$$A^{(k)}_{i+1/2} = A(w_i + s_k(w_{i+1} - w_i)), \quad k = 1, \ldots, q,$$

where A is the Jacobian matrix of F, and let $\lambda^{(k)}_{i+1/2,\max}$ be the eigenvalue of $A^{(k)}_{i+1/2}$ with maximum absolute value. Then, the *polynomial approximate Osher-Solomon*

flux is given by (3) with viscosity matrix

$$Q_{i+1/2} = \sum_{k=1}^{q} \omega_k \widetilde{P}^{(k)}_{i+1/2}, \tag{10}$$

where

$$\widetilde{P}^{(k)}_{i+1/2} = |\lambda^{(k)}_{i+1/2,\max}| P\left(|\lambda^{(k)}_{i+1/2,\max}|^{-1} A^{(k)}_{i+1/2} \right). \tag{11}$$

Remark 1 The advantage of formula (10) with respect to (8) is that in the latter it is necessary to compute the full eigenstructure of the system, while in the former only an upper bound on the spectral radius is needed.

Notice that the closer the polynomial $P(x)$ is to $|x|$ in the uniform norm, the more similar the approximate flux (10) will be to the Osher-Solomon flux (8). This suggests to use accurate polynomial approximations to $|x|$ for building (10). In particular, Chebyshev approximations will be considered in the numerical experiments. Specifically, for a given $p \geq 1$ we take $P(x) = \tau_{2p}(x)$, where

$$\tau_{2p}(x) = \frac{2}{\pi} + \frac{4}{\pi} \sum_{j=1}^{p} \frac{(-1)^{j+1}}{(2j-1)(2j+1)} T_{2j}(x), \quad x \in [-1, 1],$$

$T_{2j}(x)$ being the Chebyshev polynomials. As it is well-known, the order of approximation of $\tau_{2p}(x)$ to $|x|$ is optimal in the $L^\infty(-1, 1)$ norm. Moreover, the recursive definition of the polynomials $T_{2k}(x)$ provides an explicit and efficient way to compute $\tau_{2p}(x)$.

As it is well-known, the order of approximation to $|x|$ can be greatly improved by using rational functions instead of polynomials. This suggests to consider *rational approximate Osher-Solomon* fluxes of the form (3) with viscosity matrix

$$Q_{i+1/2} = \sum_{k=1}^{q} \omega_k \widetilde{R}^{(k)}_{i+1/2}, \tag{12}$$

where $\widetilde{R}^{(k)}_{i+1/2}$ is defined as in (11), but taking as basis function a rational approximation $R(x)$ to $|x|$ satisfying the stability condition (9). Following [6], two different families of rational functions will be considered:

- Given a set of $r \geq 4$ distinct points $X = \{0 < x_1 < \cdots < x_r \leq 1\}$, construct the polynomial

$$p(x) = \prod_{k=1}^{r} (x + x_k).$$

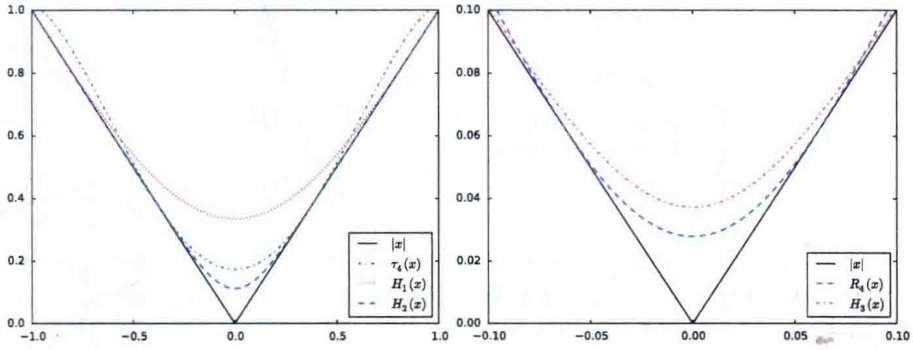

Fig. 1 *Left*: Chebyshev $\tau_4(x)$ and Halley $H_1(x)$ and $H_2(x)$ functions. *Right*: Newman $R_4(x)$ and Halley $H_3(x)$ functions. Notice the different scaling in both figures

The *Newman rational function* [10] associated to X is defined as

$$R_r(x) = x\,\frac{p(x) - p(-x)}{p(x) + p(-x)}.$$

The rate of approximation of $R_r(x)$ to $|x|$ depends on the choice of nodes X: several possibilities can be found in the literature. Here, we will take $x_k = \exp(-kr^{-1/2})$, which provides an exponential rate of approximation [10].

- The *Halley rational functions* $H_r(x)$ are recursively defined as [6]

$$H_{r+1}(x) = H_r(x)\,\frac{H_r(x)^2 + 3x^2}{3H_r(x)^2 + x^2}, \quad H_0(x) = 1.$$

It can be proved that $\|H_r(x) - |x|\|_\infty = 3^{-r}$.

Figure 1 shows a comparison between the Chebyshev $\tau_4(x)$, Newman $R_4(x)$, and Halley $H_r(x)$ $(r = 1, 2, 3)$ functions.

Both the Chebyshev polynomials $\tau_{2p}(x)$ and the Newman functions $R_r(x)$ do not satisfy the stability condition (9) strictly, although this can be easily fixed with a slight modification: see [6] for details. However, in practical computations there are no appreciable differences between both approaches. On the other hand, Halley functions $H_r(x)$ satisfy (9) by construction. As long as the functions considered do not cross the origin, no entropy-fix is needed in the presence of sonic points.

5 Application to Ideal Magnetohydrodynamics

In this section we apply the approximate Osher-Solomon schemes introduced previously to solve some challenging problems related to the ideal magnetohydrodynamics equations.