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The Dynamical Mordell–Lang Conjecture

To Jason's wife, Jessica, and kids, Chris and Caitlin

To Dragos' mother, Lidia

To Tom's wife, Amanda

Preface

This book originated from the authors' desire to give an explanation of several recent applications of p-adic analysis to number theory and especially to arithmetic geometry. Central to this end has been the work done by several people (including the authors) to prove the *Dynamical Mordell-Lang conjecture*, which gives predictions about how the orbits of points in a variety under self-maps should intersect subvarieties. As the name suggests, this can be interpreted as a dynamical analogue of the classical Mordell-Lang Conjecture (proved by Faltings and Vojta) concerning intersections between finitely generated subgroups and subvarieties in a semiabelian variety.

Many results working towards this conjecture have used p-adic analysis, and we describe all known (to us) partial results up to this point in time—both those using p-adic analysis and those using alternative approaches—towards the Dynamical Mordell-Lang Conjecture. In some cases, we present entire proofs of results, while in other cases only a sketch is given, and in certain cases only a brief overview of the idea of the proof is provided. Our choice should not be interpreted as our opinion about the relative importance of the included results, but is instead an editorial choice regarding which material we thought best fits the overarching theme of this book.

We also give other applications of p-adic analysis to number theory and arithmetic geometry. In these cases, our list of applications is not meant to be exhaustive, but rather our goal is to show the wide reach of applications and potential applications of p-adic analysis to arithmetic geometry. While the uses of p-adic analytic methods we give do not always explicitly relate to the Dynamical Mordell-Lang Conjecture, we have generally favored applications of p-adic analysis to problems with some relation to the Dynamical Mordell-Lang Conjecture.

We thank all our colleagues with whom we wrote many of the papers whose results are detailed in this book; obviously, without the joint efforts we put towards solving the Dynamical Mordell-Lang Conjecture we would not have had a topic for this book. So, we thank Rob Benedetto, Ben Hutz, Par Kurlberg, Jeff Lagarias, Tom Scanlon, Yu Yasufuku, Umberto Zannier, and Mike Zieve. We are also grateful to the referees for their careful reading of a previous version of this book, and for suggesting many improvements for our work. Last, but definitely not least, we thank our families for their love and support while writing this book.

Notation

We let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of integer, rational, real, respectively, complex numbers. $\mathbb{N}_0$ is the set of all nonnegative integers, while $\mathbb{N}$ is the set of all positive integers.

An *arithmetic progression* is a set of the form $\{a+rn\}_{n\in\mathbb{N}_0}$, where the *common difference* r may be equal to 0 (in which case the set consists of a single element). If the common difference r is nonzero, then the arithmetic progression is infinite. Note that in the literature, sometimes one calls such a sequence a *one-sided* arithmetic progression in order to distinguish it from a *two-sided arithmetic progression*, which is a set of the form $\{a+rn\}_{n\in\mathbb{Z}}$. However, since in this book we mainly encounter one-sided arithmetic progressions and only occasionally encounter two-sided arithmetic progressions, our convention is to call arithmetic progression a sequence $\{a+rn\}_{n\in\mathbb{N}_0}$, while a sequence $\{a+rn\}_{n\in\mathbb{Z}}$ is called a two-sided arithmetic progression.

For a matrix A, we denote by A^t its transpose.

For a set U, we denote by id_U the identity function on U.

For any field K, we denote by $\mathrm{char}(K)$ its characteristic. By $\overline{K}$ we denote a fixed algebraic closure of K.

For any subfield $K\subseteq\overline{\mathbb{Q}}$, we denote by $\mathfrak{o}_K$ the ring of algebraic integers contained in K. If K is a number field, and $\mathfrak{p}$ is a prime ideal of K, then $k_{\mathfrak{p}}$ is the residue field corresponding to $\mathfrak{p}$, i.e., $k_{\mathfrak{p}}\xrightarrow{\sim}\mathfrak{o}_K/\mathfrak{p}$.

The usual affine space of dimension m is denoted by $\mathbb{A}^m$; for any field K, we have that $\mathbb{A}^m(K)$ consists of all m-tuples of points with coordinates in K. Similarly, we denote by $\mathbb{P}^m$ the projective space of dimension m; for any field K, we have that $\mathbb{P}^m(K)$ consists of all equivalence classes of $(m+1)$-tuples of points with coordinates in K not all equal to 0, under the equivalence relation

$$[x_0 : x_1 \cdots : x_m] \sim [y_0 : y_1 : \cdots : y_m]$$

if and only if there exists a nonzero scalar $c\in K$ such that

$$y_i = cx_i \text{ for all } i=1,\dots,m.$$

By *affine variety* we mean a subset of an affine space defined by a set of algebraic equations. Note that we do not ask *a priori* the variety be irreducible. Similarly, by *projective variety* we mean a subset of a projective space defined by a set of algebraic equations. We endow both the affine space and the projective space with the Zariski topology where the closed sets are precisely the (affine, respectively projective) varieties. We say that X is a quasiprojective variety if it is the open subset of a projective subvariety of some projective space. We say that a variety X is defined over a field K if it may be defined by a set of equations with coefficients in K. For a variety X defined over a field K, we denote by $X(K)$ the set of K-rational points of X.

We denote by $\mathbb{G}_a$ the affine line $\mathbb{A}^1$ endowed with the additive group law; we extend this law coordinatewise to $\mathbb{G}_a^n$. We denote by $\mathbb{G}_m$ the (Zariski open subset of the affine line) $\mathbb{A}^1\setminus\{0\}$, i.e., the affine line without the origin, endowed with the multiplicative group law. Similarly to $\mathbb{G}_a^n$, we extend the multiplicative group law to $\mathbb{G}_m^n$.

An *abelian variety* is an irreducible projective variety which has the structure of an algebraic group.

For a set X, a map $\Phi : X \longrightarrow X$ is called a *self-map*. In general, for a self-map $\Phi : X \longrightarrow X$ and for any integer $n \geq 0$, we denote by Φ^n the n-th compositional iterate of Φ, i.e. $\Phi^n = \Phi \circ \cdots \circ \Phi$ (n times), with the convention that Φ^0 is the identity map. The *orbit* of a point $x \in X$ is denoted as $\mathcal{O}_\Phi(x)$ and it is the set of all $\Phi^n(x)$ for $n \in \mathbb{N}_0$.

A *dynamical system* consists of a topological space X endowed with a continuous self-map Φ.

For two real-valued functions f and g, we write $f(x) = o(g(x))$ if $\lim_{x\to\infty} f(x)/g(x) = 0$. Similarly, we write $f(x) = O(g(x))$ if the function $x \mapsto f(x)/g(x)$ is bounded as $x \to \infty$.

In a metric space $(X, d(\cdot,\cdot))$, for $x \in X$ and $r \in \mathbb{R}_{>0}$ we denote by $D(x, r)$ the open disk

$$D(x, r) = \{y \in X : d(x, y) < r\}.$$

We denote by $\overline{D}(x, r)$ the closure of $D(x, r)$.

Contents

CHAPTER 1

Introduction

In this chapter we describe various instances of the Dynamical Mordell-Lang Conjecture which appear in seemingly different areas. We conclude our Introduction by giving a brief overview of the rest of the book.

1.1. Overview of the problem

We start by presenting several arithmetic questions which are all connected, though this may not be so obvious *a priori*. All these questions have in common the following theme: we have a dynamical system Φ on a topological space X, and then for a point $\alpha \in X$ and a closed subset V of X, we ask for what values of $n \in \mathbb{N}_0$ we have $\Phi^n(\alpha) \in V$? The underlying theme of this book is that all the questions we consider have, or are conjectured to have, the same answer to the above question: *finitely many arithmetic progressions*. We also recall our convention that an arithmetic progression of common difference equal to 0 is simply a *singleton*.

The cases we consider are the following ones:

(1) Find all $n \in \mathbb{N}_0$ such that $a_n = 0$ where $\{a_n\}_{n\in\mathbb{N}_0}$ is a linear recurrence sequence. Say that the recurrence relation verified by the sequence is given for all $n \geq 0$ by

$$a_{n+m} = c_1 a_{n+m-1} + \cdots + c_m a_n,$$

for some given complex numbers $c_1, \ldots, c_m$. Then the ambient space is the affine space $\mathbb{A}^m$ with the Zariski topology, while the dynamical system is the one given by

$$\Phi\left((x_1, \ldots, x_m)\right) = (x_2, \ldots, x_m, c_1 x_m + \cdots + c_m x_1),$$

the starting point of the iteration is

$$x := (a_0, \ldots, a_{m-1}),$$

and $V \subset \mathbb{A}^m$ is the hyperplane given by the equation $x_1 = 0$. In Section 1.2 and Subsection 2.5.1, we explain this example in greater detail. In Section 2.5 we prove that the answer to this question is always a *finite union of arithmetic progressions*. A related, but more general problem involving (multi-dimensional) polynomial-exponential equations is discussed in Section 1.3.

(2) Find all $n \in \mathbb{N}_0$ such that given a matrix $A \in M_n(\mathbb{C})$ acting on the complex affine space $\mathbb{A}^n(\mathbb{C})$, a point $\alpha \in \mathbb{A}^n(\mathbb{C})$, and a subvariety $V \subset \mathbb{A}^n$, then $A^n\alpha \in V(\mathbb{C})$. This case is discussed in Section 1.4 and it turns out to be equivalent with the problem (1) discussed above (see the equivalence proven in Proposition 2.5.1.4).

(3) Find all $n \in \mathbb{N}_0$ such that given an endomorphism Φ of a quasiprojective variety X defined over $\mathbb{C}$, a point $\alpha \in X(\mathbb{C})$, and a subvariety V of X, then $\Phi^n(x) \in V(\mathbb{C})$. This problem, called the *Dynamical Mordell-Lang Conjecture* generalizes both of the above problems described above (see Section 1.5 for a first discussion of this conjecture). It is expected the answer to this question is again *finitely many arithmetic progressions.*

(4) Given a power series

$$f(z) := \sum_{n=0}^{\infty} a_n z^n$$

which satisfies a linear differential equation with polynomial coefficients, describe the set

$$S_f := \{n \in \mathbb{N}_0 \colon a_n = 0\}.$$

Rubel [**Rub83**, Problem 16] conjectured that S_f is a finite union of arithmetic progressions. We discuss this problem in Subsection 3.2.1, and show that a positive answer for an extension of the above Dynamical Mordell-Lang Conjecture to rational maps would solve Rubel's question.

1.2. Linear recurrence sequences

Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence defined by

$$F_0 = 0,\ F_1 = 1 \text{ and } F_{n+2} = F_{n+1} + F_n \text{ for all } n \geq 0.$$

Also, let $\{a_n\}_{n\geq 0}$ be the sequence defined recursively by

$$a_{n+2} = 5a_{n+1} - 6a_n,$$

where $a_0 = \frac{7}{12}$ and $a_1 = \frac{3}{2}$.

QUESTION 1.2.0.1. *What are the numbers which appear in both of the sequences* $\{F_m\}_{m\in\mathbb{N}_0}$ *and* $\{a_n\}_{n\in\mathbb{N}_0}$*?*

We can compute easily the first elements in both sequences:

$$F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ F_5 = 5,\ F_6 = 8,\ F_7 = 13,\ \ldots$$

and

$$a_0 = \frac{7}{12},\ a_1 = \frac{3}{2},\ a_2 = 4,\ a_3 = 11,\ a_4 = 31,\ a_5 = 89,\ a_6 = 259\ldots.$$

One observes that $F_{11} = 89 = a_5$, and it is a reasonable question to ask whether this is the only answer to Question 1.2.0.1. This is a hard question since one would have to solve the equation $F_m = a_n$ in nonnegative integers m and n (for more details, see [**Eve95**]). Moreover, since it is easy to find a formula for the general term of both of these sequences (see Proposition 2.5.1.4), Question 1.2.0.1 reduces to finding $m, n \in \mathbb{N}$ such that

$$\frac{1}{\sqrt{5}} \cdot \left(\left(\frac{1+\sqrt{5}}{2} \right)^m - \left(\frac{1-\sqrt{5}}{2} \right)^m \right) = 2^{n-2} + 3^{n-1}.$$

On the other hand, if we were to ask the easier question of when the above equality holds when $m = n$, the answer would be *never* since (by a simple inductive argument) one can show that $a_k > F_k$ for all $k \in \mathbb{N}$.

In general, given two linear recurrence sequences $\{a_m\}_{m\in\mathbb{N}_0}$ and $\{b_n\}_{n\in\mathbb{N}_0}$, one would like to understand whether there exists an underlying *structure* for the solutions $(m,n)\in\mathbb{N}\times\mathbb{N}$ for which $a_m=b_n$. Or, at least, for the easier case, one would like to understand the structure of the set of all $n\in\mathbb{N}$ such that $a_n=b_n$. It is immediate to see that this last case reduces to understanding when a given linear recurrence sequence $\{c_n\}_{n\in\mathbb{N}_0}$ (in this case, $c_n=a_n-b_n$) takes the value 0. Then the answer is that if there exist infinitely many $n\in\mathbb{N}$ such that $c_n=0$, then there exists an *infinite* arithmetic progression $\{\ell+nk\}_{n\in\mathbb{N}_0}$ such that $c_{\ell+nk}=0$. This will be proven in Section 2.5. As described in Section 1.1, the proper dynamical setting for this example is as follows: given a linear recurrence sequence

$$\{a_m\}_{m\in\mathbb{N}_0}\subset\mathbb{C}$$

which satisfies the relation

$$a_{n+m}=c_1a_{n+m-1}+\cdots+c_ma_n,$$

for some given complex numbers $c_1,\dots,c_m$, then the dynamical system is the one given by the map

$$\Phi((x_1,\dots,x_m))=(x_2,\dots,x_m,c_1x_m+\cdots+c_mx_1)$$

acting on the m-dimensional affine complex space $\mathbb{A}^m$. Then finding all $n\in\mathbb{N}_0$ such that $a_n=0$ is equivalent with finding all $n\in\mathbb{N}_0$ such that

$$\Phi^n((a_0,\dots,a_{m-1}))\in V(\mathbb{C}),$$

where $V\subset\mathbb{A}^m$ is the hyperplane given by the equation $x_1=0$.

1.3. Polynomial-exponential Diophantine equations

Let $m,k\in\mathbb{N}$, let $F\in\mathbb{Z}[x_1,\dots,x_m,y_1,\dots,y_k]$, and let $r_1,\dots,r_k\in\mathbb{Z}$. A polynomial-exponential equation has the form

$$F\left(j_1,\dots,j_m;r_1^{n_1},\dots,r_k^{n_k}\right)=0,$$

where the variables $j_1,\dots,j_m\in\mathbb{Z}$, respectively $r_1,\dots,r_k\in\mathbb{N}_0$. In general, there might be *many* solutions to the above equation, especially if the degree of f in x_i is 1 for at least one variable x_i. But, even if $\deg_{x_i}f=1$, there might be no solutions due to some local constraints such as in the following case:

$$21x_1^3x_3-7\cdot 3^{n_1}x_2+14\cdot 5^{n_2}x_3^2-49x_1x_3+2=0, \tag{1.3.0.1}$$

when there are no solutions $x_1,x_2,x_3\in\mathbb{Z}$ and $n_1,n_2\in\mathbb{N}_0$ by considering the congruence modulo 7 for the equation (1.3.0.1). Now, even if one assumes $j_1=j_2=\cdots=j_m=j$, and that the polynomial f has the variables x_i and y_j separated, the problem is not easier. Even also assuming that $r_1=r_2\cdots=r_k$ does not simplify the problem much. For example, we discuss in Chapter 13 the following special case:

$$g(x)=\sum_{i=1}^k c_ip^{n_i},$$

where $g\in\mathbb{Z}[x]$, $c_1,\dots,c_k\in\mathbb{Z}$ and p is a prime number. Essentially, one *expects* that if $g(x)$ has *few* nonzero p-adic digits, then x (or a linear function evaluated at x) would also have *few* p-adic digits. However, this is far from being proven even in simple cases such as $g(x)=x^2$ and $k\geq 5$ (for more details, see [**BBM13, CZ00, CZ13**] and the references therein).

On the other hand, if one assumes that

$$j_1 = \cdots = j_m = n_1 = n_2 = \cdots = n_k,$$

then the problem reduces essentially to the one discussed in Section 1.2 (see also Section 2.5). Thus one obtains that if there exist infinitely many $n \in \mathbb{Z}$ such that

$$H\left(n, r_1^n, r_2^n \ldots, r_k^n\right) = 0, \tag{1.3.0.2}$$

where $H \in \mathbb{Z}[z_0, z_1, z_2, \ldots, z_k]$, then there exists an infinite arithmetic progression $\{\ell + nk\}_{n\in\mathbb{N}_0}$ such that each element of it is a solution to (1.3.0.2).

1.4. Linear algebra

Let A be an invertible matrix in $\mathrm{GL}_r(\mathbb{C})$, let V be a linear subspace of $\mathbb{C}^r$, and let $z \in \mathbb{C}^r$.

QUESTION 1.4.0.1. *Is there a simple description of the set of positive integers n such that $A^n z \in V$?*

We note that the problem discussed in this section could easily be asked for an arbitrary subvariety defined over $\mathbb{C}$ of the affine space $\mathbb{A}^r$; however this more general question reduces to the case V is a linear subvariety.

If V is a line passing through the origin of $\mathbb{C}^r$, then once there exist two distinct nonnegative integers $m < n$ such that

$$A^m z \in V \text{ and } A^n z \in V, \tag{1.4.0.2}$$

then we immediately conclude that V is fixed by A^{n-m} and therefore

$$A^{m+\ell(n-m)} z \in V \text{ for all } \ell \in \mathbb{N}_0.$$

In particular, if k_0 is the smallest positive integer k such that A^k fixes V, and if m_0 is the smallest nonnegative integer m such that $A^m z \in V$, then $A^n z \in V$ if and only if $n = m_0 + \ell k_0$ for some nonnegative integer ℓ.

Things are not so simple in general. For example, when V is a line that does not pass through the origin, it is easy to see that you can have distinct m and n such that (1.4.0.2) holds without getting an entire arithmetic progression of such integers, just by choosing a line V which passes through two arbitrary points $A^m z$ and $A^n z$. But in the case of lines not passing through the origin, once you have a large finite number of integers n such that

$$A^n z \in V, \tag{1.4.0.3}$$

you must have an infinite arithmetic progression of such n. There is even an explicit bound on that number due to Beukers-Schlickewei [**BS96**], which is likely nowhere near sharp. In fact, under the assumption that each eigenvalue of A is either equal to 1 or is not a root of unity, and furthermore for each two distinct eigenvalues λ_i and λ_j of A we have that λ_i/λ_j is not a root of unity, Beukers-Schlickewei [**BS96**] show that there are at most 61 integers $n \in \mathbb{N}_0$ such that (1.4.0.3) holds. The general case of an arbitrary matrix A follows easily from this special case.

1.5. Arithmetic geometry

The subject of this book is a geometric generalization (see Conjecture 1.5.0.1) of all of the above problems, and it is also connected to the classical Mordell-Lang Conjecture (see Chapter 3). In each of the three problems discussed in Sections 1.2 to 1.4, we deal with a geometric object: the line V in Section 1.4, or the hypersurface $F = 0$ in Section 1.3, or the hyperplane $x_1 = 0$ in the affine space $\mathbb{A}^m$ as in Section 1.2. And we want to understand when an arithmetic dynamical system intersects the geometric object. The arithmetic dynamical system is the iteration of the matrix A in Section 1.4, or the input of an integer number into the equation $F = 0$ (which is a discrete dynamical system simply because all integers are obtained from 0 by repeated operations of either $z \mapsto z+1$ or $z \mapsto z-1$), or a linear recurrence sequence as in Section 1.2. And in each case one obtains that once there exist infinitely many instances of the intersection between the geometric object and the arithmetic dynamical object, then there is a *structure* for the intersection which is given by *finitely many arithmetic progressions*. This principle is formally stated in the Dynamical Mordell-Lang Conjecture (for more details, see Chapter 3).

CONJECTURE 1.5.0.1 (Dynamical Mordell-Lang Conjecture). *Let X be a quasi-projective variety defined over $\mathbb{C}$, let Φ be any endomorphism of X, let $\alpha \in X(\mathbb{C})$, and let $V \subseteq X$ be any subvariety. Then the set of all $n \in \mathbb{N}_0$ such that $\Phi^n(\alpha) \in V(\mathbb{C})$ is a union of finitely many arithmetic progressions.*

We note that the Dynamical Mordell-Lang Conjecture can be formulated over any field K of characteristic 0 (see Conjecture 3.1.1.1); however such a formulation reduces to proving the case when $K = \mathbb{C}$ (see Proposition 3.1.2.1).

A special case of Conjecture 1.5.0.1 that is known is when X is an abelian variety, and Φ is the translation-by-P endomorphism of X for some point $P \in X(\mathbb{C})$. In this latter case we encounter the cyclic case of the classical Mordell-Lang Conjecture (for more details, see Section 3.4).

We present below a few cases of Conjecture 1.5.0.1; all our examples are set in the ambient space $X = \mathbb{A}^3$ in which case there is at this time no general proof of the Dynamical Mordell-Lang Conjecture.

EXAMPLE 1.5.0.2. Consider the endomorphism

$$\Phi : \mathbb{A}^3 \longrightarrow \mathbb{A}^3$$

given by

$$\Phi(x, y, z) = (x^2 + x, y^2 + y, z^2 + z).$$

Let $V \subset \mathbb{A}^3$ be the plane given by the equation

$$x + y + z = 1.$$

Then for *most* points $\alpha \in \mathbb{A}^3(\overline{\mathbb{Q}})$, the set

$$S := \{n \in \mathbb{N}_0 \colon \Phi^n(\alpha) \in V(\overline{\mathbb{Q}})\}$$

is finite. For example, this can be seen immediately if all three coordinates of α are integers (in which case, at the very most, S has 1 element). However, if α is an arbitrary point in $\mathbb{A}^3(\overline{\mathbb{Q}})$, then it is much harder to prove that S is always a finite union of arithmetic progressions (possibly with common difference equal to 0). However, we will see later (see Corollary 7.0.0.1) that for *any* subvariety

$V \subseteq \mathbb{A}^3$, the set S is a finite union of artihmetic progressions. Furthermore, using the classification of periodic curves under the coordinatewise action of a polynomial done by Medvedev-Scanlon [**MS14**], one can show that in the case of the above plane V, the set S is finite assuming α is not preperiodic. Now, if α is preperiodic, the question of whether S contains an infinite arithmetic progression is equivalent with finding three preperiodic points a, b and c for the action of the polynomial

$$f(z) := z^2 + z,$$

such that

$$a + b + c = 1.$$

This last question is a deep question related to the problem of unlikely intersections in dynamics which we discuss in Subsection 14.2.2.

EXAMPLE 1.5.0.3. Consider the endomorphism

$$\Phi : \mathbb{A}^3 \longrightarrow \mathbb{A}^3$$

given by

$$\Phi(x, y, z) = (x^5, y^3, z^5).$$

Let $\alpha = (0, i, 0)$ and let $V \subset \mathbb{A}^3$ be the surface given by the equation

$$x^3 + y + z^3 = i.$$

We easily see that α is periodic under the action of Φ and moreover, $\Phi^n(\alpha) \in V$ if and only if n is an even nonnegative integer. Actually, using Theorem 9.3.0.1 one can show that for any $\alpha \in \mathbb{A}^3(\mathbb{C})$ and for any complex subvariety $V \subseteq \mathbb{A}^3$, the set S of all $n \in \mathbb{N}_0$ such that $\Phi^n(\alpha) \in V(\mathbb{C})$ is a finite union of arithmetic progressions. Furthermore, according to the classical Mordell-Lang conjecture for an algebraic torus (proven by Laurent [**Lau84**]; see also Section 3.4), one obtains that the above set S is finite unless V contains a translate of a positive dimensional algebraic torus.

EXAMPLE 1.5.0.4. Consider the endomorphism

$$\Phi : \mathbb{A}^3 \longrightarrow \mathbb{A}^3$$

given by

$$\Phi(x, y, z) = (x^2 + y, y^2 + z, z^2 + x).$$

Let $\alpha = (1, 1, 1)$ and $S \subset \mathbb{A}^3$ be the surface given by the equation

$$x + y^2 + z^3 = x^2 + y^3 + z.$$

It is immediate to see that the entire orbit $\mathcal{O}_\Phi(\alpha)$ is contained in the surface S, and the reason for this is that V contains the line L given by the equation

$$x = y = z,$$

which is fixed by the action of Φ. However, if V is an arbitrary subvariety of $\mathbb{A}^3$, and also α is an arbitrary point in $\mathbb{A}^3(\mathbb{C})$, then it is not known whether Conjecture 1.5.0.1 holds. In some sense, the endomorphism Φ from this Example lies outside all the presently known cases of the Dynamical Mordell-Lang Conjecture (see Chapter 3 for more details).