Graduate Texts in Mathematics

James E. Humphreys

Introduction to Lie Algebras and Representation Theory

李代数和表示论导论

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内容简介

本书是一部优秀的李群及其表示论研究生教材，深受数学专业和物理专业的研究生好评。本书初版于1972年，以后经过多次修订重印，本书是1997年的第7次修订重印版。书中对一些问题的处理很有特色，立足点较高，但叙述十分清晰，如线性变换的Jordan-Chevalley分解、Cartan子代数的共轭定理、同构定理的证明、根系统的公理化处理、Weyl特征子公式、Chevalley群的基本结构等。

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Preface

This book is designed to introduce the reader to the theory of semisimple Lie algebras over an algebraically closed field of characteristic 0, with emphasis on representations. A good knowledge of linear algebra (including eigenvalues, bilinear forms, euclidean spaces, and tensor products of vector spaces) is presupposed, as well as some acquaintance with the methods of abstract algebra. The first four chapters might well be read by a bright undergraduate; however, the remaining three chapters are admittedly a little more demanding.

Besides being useful in many parts of mathematics and physics, the theory of semisimple Lie algebras is inherently attractive, combining as it does a certain amount of depth and a satisfying degree of completeness in its basic results. Since Jacobson's book appeared a decade ago, improvements have been made even in the classical parts of the theory. I have tried to incorporate some of them here and to provide easier access to the subject for non-specialists. For the specialist, the following features should be noted:

(1) The Jordan-Chevalley decomposition of linear transformations is emphasized, with "toral" subalgebras replacing the more traditional Cartan subalgebras in the semisimple case.

(2) The conjugacy theorem for Cartan subalgebras is proved (following D. J. Winter and G. D. Mostow) by elementary Lie algebra methods, avoiding the use of algebraic geometry.

(3) The isomorphism theorem is proved first in an elementary way (Theorem 14.2), but later obtained again as a corollary of Serre's Theorem (18.3), which gives a presentation by generators and relations.

(4) From the outset, the simple algebras of types A, B, C, D are emphasized in the text and exercises.

(5) Root systems are treated axiomatically (Chapter III), along with some of the theory of weights.

(6) A conceptual approach to Weyl's character formula, based on Harish-Chandra's theory of "characters" and independent of Freudenthal's multiplicity formula (22.3), is presented in §23 and §24. This is inspired by D.-N. Verma's thesis, and recent work of I. N. Bernstein, I. M. Gel'fand, S. I. Gel'fand.

(7) The basic constructions in the theory of Chevalley groups are given in Chapter VII, following lecture notes of R. Steinberg.

I have had to omit many standard topics (most of which I feel are better suited to a second course), e.g., cohomology, theorems of Levi and Mal'cev, theorems of Ado and Iwasawa, classification over non-algebraically closed fields, Lie algebras in prime characteristic. I hope the reader will be stimulated to pursue these topics in the books and articles listed under References, especially Jacobson [1], Bourbaki [1], [2], Winter [1], Seligman [1].

A few words about mechanics: Terminology is mostly traditional, and notation has been kept to a minimum, to facilitate skipping back and forth in the text. After Chapters I–III, the remaining chapters can be read in almost any order if the reader is willing to follow up a few references (except that VII depends on §20 and §21, while VI depends on §17). A reference to Theorem 14.2 indicates the (unique) theorem in subsection 14.2 (of §14). Notes following some sections indicate nonstandard sources or further reading, but I have not tried to give a history of each theorem (for historical remarks, cf. Bourbaki [2] and Freudenthal-deVries [1]). The reference list consists largely of items mentioned explicitly; for more extensive bibliographies, consult Jacobson [1], Seligman [1]. Some 240 exercises, of all shades of difficulty, have been included; a few of the easier ones are needed in the text.

This text grew out of lectures which I gave at the N.S.F. Advanced Science Seminar on Algebraic Groups at Bowdoin College in 1968; my intention then was to enlarge on J.-P. Serre's excellent but incomplete lecture notes [2]. My other literary debts (to the books and lecture notes of N. Bourbaki, N. Jacobson, R. Steinberg, D. J. Winter, and others) will be obvious. Less obvious is my personal debt to my teachers, George Seligman and Nathan Jacobson, who first aroused my interest in Lie algebras. I am grateful to David J. Winter for giving me pre-publication access to his book, to Robert L. Wilson for making many helpful criticisms of an earlier version of the manuscript, to Connie Engle for her help in preparing the final manuscript, and to Michael J. DeRise for moral support. Financial assistance from the Courant Institute of Mathematical Sciences and the National Science Foundation is also gratefully acknowledged.

New York, April 4, 1972 J. E. Humphreys

Notation and Conventions

Z, $\mathbf{Z}^+$, **Q**, **R**, **C** denote (respectively) the integers, nonnegative integers, rationals, reals, and complex numbers

∐ denotes direct sum of vector spaces

$A \rtimes B$ denotes the semidirect product of groups A and B, with B normal

Card	= cardinality	Ker	= kernel
char	= characteristic	Im	= image
det	= determinant	Tr	= trace
dim	= dimension		

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Chapter I

Basic Concepts

In this chapter F *denotes an arbitrary (commutative) field.*

1. Definitions and first examples

1.1. The notion of Lie algebra

Lie algebras arise "in nature" as vector spaces of linear transformations endowed with a new operation which is in general neither commutative nor associative: $[x, y] = xy - yx$ (where the operations on the right side are the usual ones). It is possible to describe this kind of system abstractly in a few axioms.

Definition. A vector space L over a field F, with an operation $L \times L \to L$, denoted $(x, y) \mapsto [xy]$ and called the **bracket** or **commutator** of x and y, is called a **Lie algebra** over F if the following axioms are satisfied:

($L1$) The bracket operation is bilinear.
($L2$) $[xx] = 0$ for all x in L.
($L3$) $[x[yz]] + [y[zx]] + [z[xy]] = 0 \quad (x, y, z \in L)$.

Axiom ($L3$) is called the **Jacobi identity.** Notice that ($L1$) and ($L2$), applied to $[x+y, x+y]$, imply anticommutativity: ($L2'$) $[xy] = -[yx]$. (Conversely, if char F $\neq 2$, it is clear that ($L2'$) will imply ($L2$).)

We say that two Lie algebras L, L' over F are **isomorphic** if there exists a vector space isomorphism ϕ: $L \to L'$ satisfying $\phi([xy]) = [\phi(x)\phi(y)]$ for all x, y in L (and then ϕ is called an **isomorphism** of Lie algebras). Similarly, it is obvious how to define the notion of (Lie) **subalgebra** of L: A subspace K of L is called a subalgebra if $[xy] \in K$ whenever $x, y \in K$; in particular, K is a Lie algebra in its own right relative to the inherited operations. Note that any nonzero element $x \in L$ defines a one dimensional subalgebra Fx, with trivial multiplication, because of ($L2$).

In this book we shall be concerned almost exclusively with Lie algebras L whose underlying vector space is *finite dimensional* over F. *This will always be assumed, unless otherwise stated.* We hasten to point out, however, that certain infinite dimensional vector spaces and associative algebras over F will play a vital role in the study of representations (Chapters V–VII). We also mention, before looking at some concrete examples, that the axioms for a Lie algebra make perfectly good sense if L is only assumed to be a module over a commutative ring, but we shall not pursue this point of view here.

1.2. Linear Lie algebras

If V is a finite dimensional vector space over F, denote by End V the set of linear transformations $V \to V$. As a vector space over F, End V has dimension n^2 ($n = \dim V$), and End V is a ring relative to the usual product operation. Define a new operation $[x, y] = xy - yx$, called the **bracket** of x and y. With this operation End V becomes a Lie algebra over F: axioms ($L1$) and ($L2$) are immediate, while ($L3$) requires a brief calculation (which the reader is urged to carry out at this point). In order to distinguish this new algebra structure from the old associative one, we write $\mathfrak{gl}(V)$ for End V viewed as Lie algebra and call it the **general linear algebra** (because it is closely associated with the **general linear group** $GL(V)$ consisting of all invertible endomorphisms of V). When V is infinite dimensional, we shall also use the notation $\mathfrak{gl}(V)$ without further comment.

Any subalgebra of a Lie algebra $\mathfrak{gl}(V)$ is called a **linear Lie algebra**. The reader who finds matrices more congenial than linear transformations may prefer to fix a basis for V, thereby identifying $\mathfrak{gl}(V)$ with the set of all $n \times n$ matrices over F, denoted $\mathfrak{gl}(n, \mathsf{F})$. This procedure is harmless, and very convenient for making explicit calculations. For reference, we write down the multiplication table for $\mathfrak{gl}(n, \mathsf{F})$ relative to the standard basis consisting of the matrices e_{ij} (having 1 in the (i, j) position and 0 elsewhere). Since $e_{ij}e_{kl} = \delta_{jk}e_{il}$, it follows that:

(*) $$[e_{ij}, e_{kl}] = \delta_{jk}\, e_{il} - \delta_{li}\, e_{kj}.$$

Notice that the coefficients are all ± 1 or 0; in particular, all of them lie in the prime field of F.

Now for some further examples, which are central to the theory we are going to develop in this book. They fall into four families A_ℓ, B_ℓ, C_ℓ, D_ℓ ($\ell \geq 1$) and are called the **classical algebras** (because they correspond to certain classical linear Lie groups). For $\mathsf{B}_\ell - \mathsf{D}_\ell$, let char F $\neq 2$.

A_ℓ: Let $\dim V = \ell + 1$. Denote by $\mathfrak{sl}(V)$, or $\mathfrak{sl}(\ell+1, \mathsf{F})$, the set of endomorphisms of V having trace zero. (Recall that the **trace** of a matrix is the sum of its diagonal entries; this is independent of choice of basis for V, hence makes sense for an endomorphism of V.) Since $Tr(xy) = Tr(yx)$, and $Tr(x+y) = Tr(x) + Tr(y)$, $\mathfrak{sl}(V)$ is a subalgebra of $\mathfrak{gl}(V)$, called the **special linear algebra** because of its connection with the **special linear group** $SL(V)$ of endomorphisms of det 1. What is its dimension? On the one hand $\mathfrak{sl}(V)$ is a proper subalgebra of $\mathfrak{gl}(V)$, hence of dimension at most $(\ell+1)^2 - 1$. On the other hand, we can exhibit this number of linearly independent matrices of trace zero: Take all e_{ij} ($i \neq j$), along with all $h_i = e_{ii} - e_{i+1,i+1}$ ($1 \leq i \leq \ell$), for a total of $\ell + (\ell+1)^2 - (\ell+1)$ matrices. We shall always view this as the standard basis for $\mathfrak{sl}(\ell+1, \mathsf{F})$.

C_ℓ: Let $\dim V = 2\ell$, with basis $(v_1, \ldots, v_{2\ell})$. Define a nondegenerate skew-symmetric form f on V by the matrix $s = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}$. (It can be shown that even dimensionality is a necessary condition for existence of a nondegenerate bilinear form satisfying $f(v, w) = -f(w, v)$.) Denote by $\mathfrak{sp}(V)$, or $\mathfrak{sp}(2\ell, \mathsf{F})$, the **symplectic algebra**, which by definition consists of all endomorphisms x of V satisfying $f(x(v), w) = -f(v, x(w))$. The reader can easily verify that $\mathfrak{sp}(V)$ is closed under the bracket operation. In matrix terms, the condition for $x = \begin{pmatrix} m & n \\ p & q \end{pmatrix}$ ($m, n, p, q \in \mathfrak{gl}(\ell, \mathsf{F})$) to be symplectic is that $sx = -x^t s$ (x^t = transpose of x), i.e., that $n^t = n$, $p^t = p$, and $m^t = -q$. (This last condition forces $Tr(x) = 0$.) It is easy now to compute a basis for $\mathfrak{sp}(2\ell, \mathsf{F})$. Take the diagonal matrices $e_{ii} - e_{\ell+i,\ell+i}$ $(1 \leq i \leq \ell)$, ℓ in all. Add to these all $e_{ij} - e_{\ell+j,\ell+i}$ $(1 \leq i \neq j \leq \ell)$, $\ell^2 - \ell$ in number. For n we use the matrices $e_{i,\ell+i}$ $(1 \leq i \leq \ell)$ and $e_{i,\ell+j} + e_{j,\ell+i}$ $(1 \leq i < j \leq \ell)$, a total of $\ell + \frac{1}{2}\ell(\ell-1)$, and similarly for the positions in p. Adding up, we find $\dim \mathfrak{sp}(2\ell, \mathsf{F}) = 2\ell^2 + \ell$.

B_ℓ: Let $\dim V = 2\ell + 1$ be odd, and take f to be the nondegenerate symmetric bilinear form on V whose matrix is $s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{pmatrix}$. The **orthogonal algebra** $\mathfrak{o}(V)$, or $\mathfrak{o}(2\ell+1, \mathsf{F})$, consists of all endomorphisms of V satisfying $f(x(v), w) = -f(v, x(w))$ (the same requirement as for C_ℓ). If we partition x in the same form as s, say $x = \begin{pmatrix} a & b_1 & b_2 \\ c_1 & m & n \\ c_2 & p & q \end{pmatrix}$, then the condition $sx = -x^t s$ translates into the following set of conditions: $a = 0$, $c_1 = -b_2^t$, $c_2 = -b_1^t$, $q = -m^t$, $n^t = -n$, $p^t = -p$. (As in the case of C_ℓ, this shows that $Tr(x) = 0$.) For a basis, take first the ℓ diagonal matrices $e_{ii} - e_{\ell+i,\ell+i}$ $(2 \leq i \leq \ell+1)$. Add the 2ℓ matrices involving only row one and column one: $e_{1,\ell+i+1} - e_{i+1,1}$ and $e_{1,i+1} - e_{\ell+i+1,1}$ $(1 \leq i \leq \ell)$. Corresponding to $q = -m^t$, take (as for C_ℓ) $e_{i+1,j+1} - e_{\ell+j+1,\ell+i+1}$ $(1 \leq i \neq j \leq \ell)$. For n take $e_{i+1,\ell+j+1} - e_{j+1,\ell+i+1}$ $(1 \leq i < j \leq \ell)$, and for p, $e_{i+\ell+1,j+1} - e_{j+\ell+1,i+1}$ $(1 \leq j < i \leq \ell)$. The total number of basis elements is $2\ell^2 + \ell$ (notice that this was also the dimension of C_ℓ).

$\mathsf{D}_{\ell(\ell \geq 2)}$: Here we obtain another **orthogonal algebra**. The construction is identical to that for B_ℓ, except that $\dim V = 2\ell$ is even and s has the simpler form $\begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix}$. We leave it as an exercise for the reader to construct a basis and to verify that $\dim \mathfrak{o}(2\ell, \mathsf{F}) = 2\ell^2 - \ell$ (Exercise 8).

We conclude this subsection by mentioning several other subalgebras of $\mathfrak{gl}(n, \mathsf{F})$ which play an important subsidiary role for us. Let $\mathfrak{t}(n, \mathsf{F})$ be the set of **upper triangular matrices** (a_{ij}), $a_{ij} = 0$ if $i > j$. Let $\mathfrak{n}(n, \mathsf{F})$ be the **strictly upper triangular matrices** ($a_{ij} = 0$ if $i \geq j$). Finally, let $\mathfrak{d}(n, \mathsf{F})$ be the set of all **diagonal matrices**. It is trivial to check that each of these is closed under the bracket. Notice also that $\mathfrak{t}(n, \mathsf{F}) = \mathfrak{d}(n, \mathsf{F}) + \mathfrak{n}(n, \mathsf{F})$ (vector space direct sum), with $[\mathfrak{d}(n, \mathsf{F}), \mathfrak{n}(n, \mathsf{F})] = \mathfrak{n}(n, \mathsf{F})$, hence $[\mathfrak{t}(n, \mathsf{F}), \mathfrak{t}(n, \mathsf{F})] = \mathfrak{n}(n, \mathsf{F})$, cf. Exercise 5. (If H, K are subalgebras of L, $[H\,K]$ denotes the subspace of L spanned by commutators $[xy]$, $x \in H$, $y \in K$.)

1.3. Lie algebras of derivations

Some Lie algebras of linear transformations arise most naturally as derivations of algebras. By an **F-algebra** (not necessarily associative) we simply mean a vector space $\mathfrak{A}$ over F endowed with a bilinear operation $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$, usually denoted by juxtaposition (unless $\mathfrak{A}$ is a Lie algebra, in which case we always use the bracket). By a **derivation** of $\mathfrak{A}$ we mean a linear map δ: $\mathfrak{A} \to \mathfrak{A}$ satisfying the familiar product rule $\delta(ab) = a\delta(b) + \delta(a)b$. It is easily checked that the collection Der $\mathfrak{A}$ of all derivations of $\mathfrak{A}$ is a vector subspace of End $\mathfrak{A}$. The reader should also verify that the commutator $[\delta, \delta']$ of two derivations is again a derivation (though the ordinary product need not be, cf. Exercise 11). So Der $\mathfrak{A}$ is a subalgebra of $\mathfrak{gl}(\mathfrak{A})$.

Since a Lie algebra L is an F-algebra in the above sense, Der L is defined. Certain derivations arise quite naturally, as follows. If $x \in L$, $y \mapsto [xy]$ is an endomorphism of L, which we denote ad x. In fact, ad $x \in$ Der L, because we can rewrite the Jacobi identity (using ($L2'$)) in the form: $[x[yz]] = [[xy]z] + [y[xz]]$. Derivations of this form are called **inner**, all others **outer**. It is of course perfectly possible to have ad $x = 0$ even when $x \neq 0$: this occurs in any one dimensional Lie algebra, for example. The map $L \to$ Der L sending x to ad x is called the **adjoint representation** of L; it plays a decisive role in all that follows.

Sometimes we have occasion to view x simultaneously as an element of L and of a subalgebra K of L. To avoid ambiguity, the notation $\text{ad}_L x$ or $\text{ad}_K x$ will be used to indicate that x is acting on L (respectively, K). For example, if x is a diagonal matrix, then $\text{ad}_{\mathfrak{d}(n,F)}(x) = 0$, whereas $\text{ad}_{\mathfrak{gl}(n,F)}(x)$ need not be zero.

1.4. Abstract Lie algebras

We have looked at some natural examples of linear Lie algebras. It is known that, in fact, every (finite dimensional) Lie algebra is isomorphic to some linear Lie algebra (theorems of Ado, Iwasawa). This will not be proved here (cf. Jacobson [1] Chapter VI, or Bourbaki [1]); however, it will be obvious at an early stage of the theory that the result is true for all cases we are interested in.

Sometimes it is desirable, however, to contemplate Lie algebras abstractly. For example, if L is an arbitrary finite dimensional vector space over F, we can view L as a Lie algebra by setting $[xy] = 0$ for all $x, y \in L$. Such an algebra, having trivial Lie multiplication, is called **abelian** (because in the linear case $[x, y] = 0$ just means that x and y commute). If L is any Lie algebra, with basis $x_1, \ldots, x_n$ it is clear that the entire multiplication table of L can be recovered from the **structure constants** a_{ij}^k which occur in the expressions $[x_i x_j] = \sum_{k=1}^{n} a_{ij}^k x_k$. Those for which $i \geq j$ can even be deduced from the others, thanks to ($L2$), ($L2'$). Turning this remark around, it is possible to define an abstract Lie algebra from scratch simply by specifying a set of structure constants. Naturally, not just any set of scalars $\{a_{ij}^k\}$ will do, but a moment's thought shows that it is enough to require the "obvious" identities, those implied by (*L*2) and (*L*3):

$$a_{ii}^k = 0 = a_{ij}^k + a_{ji}^k;$$
$$\sum_k (a_{ij}^k a_{k\ell}^m + a_{j\ell}^k a_{ki}^m + a_{\ell i}^k a_{kj}^m) = 0.$$

In practice, we shall have no occasion to construct Lie algebras in this artificial way. But, as an application of the abstract point of view, we can determine (up to isomorphism) all Lie algebras of dimension ≤ 2. In dimension 1 there is a single basis vector x, with multiplication table $[xx] = 0$ (*L*2). In dimension 2, start with a basis x, y of L. Clearly, all products in L yield scalar multiples of $[xy]$. If these are all 0, then L is abelian. Otherwise, we can replace x in the basis by a vector spanning the one dimensional space of multiples of the original $[xy]$, and take y to be any other vector independent of the new x. Then $[xy] = ax$ $(a \neq 0)$. Replacing y by $a^{-1}y$, we finally get $[xy] = x$. Abstractly, therefore, at most one nonabelian L exists (the reader should check that $[xy] = x$ actually defines a Lie algebra).

Exercises

1. Let L be the real vector space $\mathbf{R}^3$. Define $[xy] = x \times y$ (cross product of vectors) for $x, y \in L$, and verify that L is a Lie algebra. Write down the structure constants relative to the usual basis of $\mathbf{R}^3$.
2. Verify that the following equations and those implied by (*L*1) (*L*2) define a Lie algebra structure on a three dimensional vector space with basis (x, y, z): $[xy] = z$, $[xz] = y$, $[yz] = 0$.
3. Let $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ be an ordered basis for $\mathfrak{sl}(2, \mathsf{F})$. Compute the matrices of ad x, ad h, ad y relative to this basis.
4. Find a linear Lie algebra isomorphic to the nonabelian two dimensional algebra constructed in (1.4). [Hint: Look at the adjoint representation.]
5. Verify the assertions made in (1.2) about $\mathfrak{t}(n, \mathsf{F})$, $\mathfrak{d}(n, \mathsf{F})$, $\mathfrak{n}(n, \mathsf{F})$, and compute the dimension of each algebra, by exhibiting bases.
6. Let $x \in \mathfrak{gl}(n, \mathsf{F})$ have n *distinct* eigenvalues $a_1, \ldots, a_n$ in F. Prove that the eigenvalues of ad x are precisely the n^2 scalars $a_i - a_j$ $(1 \leq i, j \leq n)$, which of course need not be distinct.
7. Let $\mathfrak{s}(n, \mathsf{F})$ denote the **scalar matrices** (=scalar multiples of the identity) in $\mathfrak{gl}(n, \mathsf{F})$. If char F is 0 or else a prime not dividing n, prove that $\mathfrak{gl}(n, \mathsf{F}) = \mathfrak{sl}(n, \mathsf{F}) + \mathfrak{s}(n, \mathsf{F})$ (direct sum of vector spaces), with $[\mathfrak{s}(n, \mathsf{F}), \mathfrak{gl}(n, \mathsf{F})] = 0$.
8. Verify the stated dimension of D_ℓ.
9. When char $\mathsf{F} = 0$, show that each classical algebra $L = \mathsf{A}_\ell, \mathsf{B}_\ell, \mathsf{C}_\ell$, or D_ℓ is equal to $[LL]$. (This shows again that each algebra consists of trace 0 matrices.)

10. For small values of ℓ, isomorphisms occur among certain of the classical algebras. Show that A_1, B_1, C_1 are all isomorphic, while D_1 is the one dimensional Lie algebra. Show that B_2 is isomorphic to C_2, D_3 to A_3. What can you say about D_2?
11. Verify that the commutator of two derivations of an F-algebra is again a derivation, whereas the ordinary product need not be.
12. Let L be a Lie algebra and let $x \in L$. Prove that the subspace of L spanned by the eigenvectors of ad x is a subalgebra.

2. Ideals and homomorphisms

2.1. *Ideals*

A subspace I of a Lie algebra L is called an **ideal** of L if $x \in L$, $y \in I$ together imply $[xy] \in I$. (Since $[xy] = -[yx]$, the condition could just as well be written: $[yx] \in I$.) Ideals play the role in Lie algebra theory which is played by normal subgroups in group theory and by two sided ideals in ring theory: they arise as kernels of homomorphisms (2.2).

Obviously 0 (the subspace consisting only of the zero vector) and L itself are ideals of L. A less trivial example is the **center** $Z(L) = \{z \in L | [xz] = 0 \text{ for all } x \in L\}$. Notice that L is abelian if and only if $Z(L) = L$. Another important example is the **derived algebra** of L, denoted $[LL]$, which is analogous to the commutator subgroup of a group. It consists of all linear combinations of commutators $[xy]$, and is clearly an ideal.

Evidently L is abelian if and only if $[LL] = 0$. At the other extreme, a study of the multiplication table for $L = \mathfrak{sl}(n, F)$ in (1.2) ($n \neq 2$ if char $F = 2$) shows that $L = [LL]$ in this case, and similarly for other classical linear Lie algebras (Exercise 1.9).

If I, J are two ideals of a Lie algebra L, then $I + J = \{x + y | x \in I, y \in J\}$ is also an ideal. Similarly, $[IJ] = \{\Sigma x_i y_i | x_i \in I, y_i \in J\}$ is an ideal; the derived algebra $[LL]$ is just a special case of this construction.

It is natural to analyze the structure of a Lie algebra by looking at its ideals. If L has no ideals except itself and 0, and if moreover $[LL] \neq 0$, we call L **simple**. The condition $[LL] \neq 0$ (i.e., L nonabelian) is imposed in order to avoid giving undue prominence to the one dimensional algebra. Clearly, L simple implies $Z(L) = 0$ and $L = [LL]$.

Example. Let $L = \mathfrak{sl}(2, F)$, char $F \neq 2$. Take as standard basis for L the three matrices (cf. (1.2)): $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The multiplication table is then completely determined by the equations: $[xy] = h$, $[hx] = 2x$, $[hy] = -2y$. (Notice that x, y, h are eigenvectors for ad h, corresponding to the eigenvalues $2, -2, 0$. Since char $F \neq 2$, these eigenvalues are distinct.) If $I \neq 0$ is an ideal of L, let $ax + by + ch$ be an arbitrary nonzero