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BY

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1. INTRODUCTION. In this paper a study is made of open mappings of one locally connected generalized continuum onto open subsets of another such space along lines suggested by certain fundamental properties of analytic functions of a complex variable. The locally connected generalized continuum seems to provide the appropriate setting for such a study for several reasons. In the first place, the complex plane and its regions are locally connected generalized continua. Further, any region in a locally connected generalized continuum is itself a space of the same sort. Finally, this type of space reproduces itself under open mappings since local connectedness and local compactness, although not invariant under all continuous transformations, are invariant under open ones. Since the mappings generated by analytic functions are open and map regions onto regions open in the complex plane (See Stoïlor [1]), the results obtained in this setting are immediately applicable to analytic mappings.

We begin (Sections 3-5) with a discussion of the interiority (= openness) and lightness of analytic mappings and some of its consequences such as the Maximum Modulus Theorem, the Fundamental Theorem of algebra and other results concerning zeros. This is followed (Section 6) by a study of the property of expansibility of a mapping which is suggested by the property of certain entire functions of having their minimum modulus go to infinity on an expanding sequence of circles about the origin. Mappings having this property are characterized by the preservation of non-compactness for closed generalized continua. Further, when open they are necessarily *onto* mappings; and indeed, either they take each value on a compact (non-empty) set or they take each value on a non-compact set. Thus we get a partitioning of the expansive open mappings into two classes corresponding to the division of entire functions into rationals and transcendentals which reduces to precisely this classification in the case of these functions.

Next comes (Sections 7-10) a study of quasi-interiority and compactness of mappings together with their composition and factorization properties and their relation to monotoneity and quasi-monotoneity. A uniformly convergent sequence of quasi-interior mappings is shown to have a quasi-interior limit, thus validating an analog of the Weierstrass Double Series Theorem. Results on inversion of local connectedness under light open mappings are established (Sections 11-12) enabling us to determine the class of mappings for which there exist *arbitrarily large* normal regions about any point. (Any light interior mapping admits *arbitrarily small* normal regions about any point.)

The decomposition generated by an open mapping is considered next (Sections 13-14) and consequences of the continuity of such a decomposition are applied to the multiplicity function. It is shown that under fairly general conditions a mapping generating a continuous decomposition is either of finite degree of else takes each of its values infinitely many times. Expansive mappings generate such decompositions as do also functions such as e^z and cos z, which are not expansive.

Finally (Sections 15-17) uniform interiority and uniform lightness for sequences and collections of mappings are investigated and some applications and interpretations made in the case of analytic functions. In particular a certain contact is established with the well known Bloch property.

2. DEFINITIONS. NOTATION. By a *mapping* we will always mean a continuous single valued transformation $f(x)$ of one topological space A into another one B_0. The image of A will usually be called B so that we have $f(A) = B \subset B_0$. By the *inverse* $f^{-1}(Y)$ of a subset Y of B_0 we mean the set of all $x \varepsilon A$ such that $f(x) \varepsilon Y$, *whether Y is contained in B or not*. Such a mapping is *interior*

or *open* provided the image of every open set in A is open in B and is *strongly open* provided the image of every open set in A is open in B_0. It will be noted that an open mapping is strongly open if and only if B is open in B_0. A mapping $f(A) = B$ is light provided $f^{-1}(y)$ is totally disconnected for each $y \in B$.

The boundary $\bar{U} - U$ of an open set will be denoted by $\mathrm{Fr}(U)$. A *region* in any space is a connected open set in that space. A *continuum* is a compact connected metric space and a *generalized continuum* is a locally compact separable metric connected space. In all cases we will be dealing with a metric space and we use $\rho(x,y)$ for the distance function and $V_r(X)$ for the spherical neighborhood of the set X or radius r.

For other terms and notations the reader is referred to the author's book [2].

3. INTERIORITY OF ANALYTIC FUNCTIONS. There are several known ways of showing that the mapping generated by a non-constant function $w = f(z)$ analytic in a region R of the z plane into the w plane is strongly interior, (see Stoïlow [1]), as this is a ready consequence of the basic properties of complex integrals together with power series expandibility of the function. For example, Rouché's Theorem together with the fact that the zeros of an analytic function are isolated yields strong interiority as follows:

Let $z_0 \in R$ and let $C : |z - z_0| = r$ be any circle lying with its interior in R and such that $f(z) \neq w_0 = f(z_0)$ for $z \in C$. Then if w_1 satisfies $0 < |w_1 - w_0| < \rho[w_0, f(C)]$, since $|f(z) - w_0| > |w_0 - w_1|$ on C, $f(z) - w_0$ and $f(z) - w_0 + w_0 - w_1 = f(z) - w_1$ have the same number of zeros inside C. Accordingly $f(z)$ takes the value w_1 within C so that the image of the interior of C contains the region $|w - w_0| < \rho[w_0, f(C)]$.

Since strong interiority and lightness are the fundamental topological properties of non-constant analytic functions, it seems highly desirable to establish them in as elementary a fashion as possible, using a minimum of analytical machinery. In particular, one would like to avoid basing the proof on the complex integral and series expansibility and use only the existence but not the continuity of the derivative. The author knows of no such proof nor is he prepared as yet to offer any. Nevertheless, it may be of interest to indicate two procedures which seem to isolate the difficulties involved in such a proof, and this we proceed to do. In either of the methods suggested one can only obtain lightness and strong interiority of non-constant analytic functions as a special case of the theorem established by invoking either countability of the zeros of its derivative or the fact that each such zero is of finite order.

We first prove a theorem by direct elementary methods which yields the desired properties for classes of functions including the non-constant analytic ones if we invoke the properties just mentioned of such functions but which does not require a derivative and includes also functions such as $w = \bar{z}$. This is accomplished by application of a lemma which we establish first.

(3.1) Lemma. *Suppose $w = f(z)$ continuous in a region R of the complex plane and that for a given $z_0 \in R$ there exist an integer $n > 0$ and a real number $\alpha \neq 0$ such that if $z - z_0 = re^{i\theta}$, $w - w_0 = \rho e^{i\varphi}$, $(w_0 = f(z_0))$, then ρr^{-n} is bounded from 0 for r sufficiently small and $\lim_{r \to 0,\ \theta = \text{constant}} \varphi = \varphi_\theta$ uniformly in θ and for any two values θ_1 and θ_2 of θ, $\varphi_{\theta_1} - \varphi_{\theta_2} = \alpha(\theta_1 - \theta_2)$. Then every linear interval containing w_0 intersects the f image of any disc $|z - z_0| \leqq r_1$ in a point distinct from w_0.*

Proof. Let S be any disc $|z - z_0| \leqq r_1$ lying in R and let $I: \varphi = \varphi'$, $0 \leqq \rho \leqq \rho'$ be any interval with one end point at w_0. Let $\varphi_0 = \lim_{r \to 0,\ \theta = \theta_0} \varphi$ for any fixed θ_0. Define θ_k, $k = 1, 2$, by the equation

$$\alpha\ (\theta_k - \theta_0) = \varphi' - \varphi_0 + (-1)^{k}\pi/9 .$$

Then from our hypothesis, if $\varphi_k = \lim_{r \to 0,\ \theta = \theta_k} \varphi$, $k = 1, 2$, we have

$$\varphi_1 - \varphi_0 = \alpha\ (\theta_1 - \theta_0) = \varphi' - \varphi_0 - \pi/9$$

or $\qquad \varphi_1 = \varphi' - \pi/9 .$

Similarly $\qquad \varphi_2 = \varphi' + \pi/9 .$

We may now assume, in view of our hypothesis, that the radius r_1 of S is small enough so that if $r \leqq r_1$, then

(1) $\rho < \rho'$, $\rho r^{-n} > d > 0$, and $|\varphi - \varphi_\theta| < \pi/9$ for any θ. Define C to be the short arc of the boundary of S joining z_1 and z_2 where $z_k - z_0 = r_1 e^{i\theta_k}$, $k = 1, 2$. Then if $z \in C$, $z - z_0 = r_1 e^{i\theta}$, $f(z) - w_0 = \rho e^{i\varphi}$, we have $|\varphi - \varphi_\theta| < \pi/9$.

Also, assuming $|z - z_1| \leqq |z - z_2|$,

$$|\varphi_\theta - \varphi_1| = |\alpha|\ \ |\theta - \theta_1| \leqq \tfrac{1}{2}|\alpha|\ \ |\theta_2 - \theta_1| = \tfrac{1}{2}|\varphi_2 - \varphi_1| = \pi/9 ;$$

and since $|\varphi_1 - \varphi'| = \pi/9$, we get

(2) $|\varphi - \varphi'| < \pi/3.$

The same conclusion would follow similarly if $|z - z_2| \leqq |z - z_1|$. Thus $f(z) - w_0 = \rho e^{i\varphi}$ satisfies (1) and (2) so that $f(z)$ and hence $f(C)$ lies in the part W of the annulus defined by

$$\rho' > \rho > dr^n, \quad |\varphi - \varphi'| < \pi/3 .$$

Now I bisects this region W; and since the argument of $f(z_k)$ differs from φ_k by less than $\pi/9$, $k = 1, 2$, it follows that $f(z_1)$ and $f(z_2)$ lie on opposite sides of I in W. Since $f(C)$ is connected it must intersect I in a point $\neq w_0$.

Note. If $f'(z_0)$ exists and is $\neq 0$, the hypothesis in this lemma is satisfied with $n = \alpha = 1$. If $f'(z)$ exists in R but z_0 is a zero of order k of $f'(z)$, the hypothesis is satisfied with $n = \alpha = k + 1$. Also it will be noted that the conditions are satisfied at any point z_0 in the z plane by the function $w = \bar{z}$ with $n = 1$, $\alpha = -1$.

(3.2) Theorem. *If the function $w = f(z)$ satisfies the conditions of* (3.1) *at every point z_0 in R, then f is light and strongly interior on R. Further, all points of $f^{-1}(w)$, $w \in f(R)$, are isolated points of this set.*

The latter statement, and thus the lightness of f, is an immediate consequence of the condition that for any $z_0 \in R$, ρr^{-n} is bounded from zero and hence not equal to zero for $r \neq 0$ and sufficiently small.

To establish strong interiority, let $z_0 \in R$ and choose a disc S centered at z_0 so that $S \subset R$ and $f(z) \neq w_0 = f(z_0)$ on $S - z_0$. Let $F = f\,[\mathrm{Fr}(S)]$, and let U be the interior of a circle centered at w_0 which contains no point of F. Then $U \subset f(S)$. For if U contained a point w_1 not in $f(S)$, the segment $w_1 w_0$ has a first point w_2 in $f(S)$ in the order w_1, w_0. Then $w_2 = f(z_2)$ for some interior point z_2 of S, since $U \cdot F = 0$; and this is impossible by (3.1) since the segment $w_1 w_2$ intersects $f(S)$ in just w_2.

Our second method is based on the fact that at a point $z_0 \in R$ where $f'(z_0)$ exists and is $\neq 0$ the conditions in the lemma (3.1) are readily established for $n = \alpha = 1$ using only the definition of derivative (See Titchmarsh [3] for example). This gives us a property very close to strong interiority at such points z_0. Indeed, if we assume existence of $f'(z)$ and countability of its zeros in R, we get strong interiority on the complement of these zeros and lightness of the mapping in a completely elementary way. It is then possible to employ the following general extensibility theorem to obtain strong interiority on the whole region R.

(3.3) Theorem. *(Extensibility) Let A and B_0 be locally connected generalized continua such that no region in A or B_0 is separated by any compact totally disconnected set. Let $f(A) = B \subset B_0$ be light and suppose there is a set E in A such that both E and $f(E)$ are punctiform* and f is strongly interior on $A - E$. Then f is strongly interior on A.*

Proof. Let $a \in E$ and let U be an open set containing a such that $\bar{U}$ is compact and $f[\mathrm{Fr}(U)]$ does not contain $f(a)$. This last condition is possible since the transformation f is light.

Since $f(a)$ is not on $f[\mathrm{Fr}(U)]$, we can find a region $R \subset B_0$ such that $f(a) \in R$, $R \cdot f[\mathrm{Fr}(U)] = 0$, and $\bar{R}$ is compact. It is sufficient to prove that $f(U)$ contains R. If we define $W = R - R \cdot f(U \cdot E)$, then we wish to show that $W \subset f(U)$. To show this we note that W is connected; for if $f(U \cdot E)$ separated the region R, it would be possible to find a subset of $f(U \cdot E)$ closed in R which separates R. This would contradict the fact that there exists no compact totally disconnected set which separates any region in B_0. Obviously $W \cdot f(U) \neq 0$. Moreover, $W \cdot f(U)$ is open in W. For consider $y \in W$ such that $f^{-1}(y)$ intersects U. Then there exists an $x \in U - E$ such that $f(x) = y$ and by hypothesis f is strongly interior on the set $U - E$. We also have that $W \cdot f(U)$ is closed in W. For consider any point $z \in W$ which is a limit point of $W \cdot f(U) = [R - R \cdot f(U \cdot E)] f(U)$. Then z is in $f(\bar{U}) = \overline{f(U)}$ and in R. Hence $z \in f(U)$ since $R \cdot f[\mathrm{Fr}(U)] = 0$. Accordingly $W \cdot f(U) = W \subset f(U)$. Finally $f(U) \supset R$, for all points we have deleted are images of points of $U \cdot E$.

(3.31) Corollary. *If A and B_0 are planes (or manifolds of dimension ≥ 2) and the light mapping f of A into B is strongly interior on $A - E$ where E and $f(E)$ are punctiform, then f is strongly interior on E.*

Note. This theorem and corollary are closely related to a theorem in the author's book [2] and also to a result of Stoïlow in [1].

(3.4) Theorem. *If the non-constant function $w = f(z)$ is continuous in a region R of the complex plane and has a non-zero derivative at all points of $R - X$ where X is countable, then f is light and strongly interior on R.*

First, the mapping must be light. For if f were constant on a non-degenerate continuum M in R, there would be a point $z_0 \in M - M \cdot X$ where $f'(z_0)$ exists and is $\neq 0$, which clearly is impossible as the constancy of f on M would make $f'(z_0) = 0$.

We show next that at any point $z_0 \in R - X$, f is strongly interior. To this end let $w_0 = f(z_0)$ and let S be any disc in R centered at z_0 such that $f(z) \neq w_0$ on $\mathrm{Fr}(S)$. We shall show that if $F = f[\mathrm{Fr}(S)]$, any w_1 satisfying $|w_1 - w_0| < \rho[w_0, F]$ belongs to $f(S)$ and, as S can be chosen arbitrarily small, this will give strong interiority at z_0. Suppose on the contrary that some such w_1 is not in $f(S)$. Since $f(S)$ is compact, some arc α of the circle C: $w - w_0 = |w_1 - w_0|$ fails to intersect $f(S)$. Then for each $w \in \alpha$, the first point w' of $f(S)$ on the radius $w\, w_0$ of C in the order w, w_0 is the image of a point z' interior to S. As the points w' are all distinct, there are uncountably many of the points z' and hence some $z' \in R - X$. But this is impossible by (3.1), since $f'(z') \neq 0$ and the linear interval $w'\, w$ intersects $f(S)$ only in w'.

Finally, since both X and $f(X)$ are countable, it follows from (3.3) or (3.31) that f is strongly interior on R.

*A set M is punctiform provided M contains no continuum.

4. INTERIORITY OF THE MODULUS FUNCTION. CONSEQUENCES. Let $w = f(z)$ be analytic and non-constant in a region R of the complex plane. We define $m(z) = |f(z)|$, $z \in R$, and call $m(z)$ the *modulus function*. Obviously it is continuous. As an immediate consequence of the strong interiority of $f(z)$ on R we have

Theorem. *The mapping $m(z)$ of R into the non-negative real axis is strongly interior.*

This yields at once the following for the most part standard results.

(1) *For any $a \in R$ and any neighborhood V of a in R, there exists $z \in V$ such that $m(z) > m(a)$; and if $m(a) \neq 0$, there exists $z_1 \in V$ such that $m(z_1) < m(a)$.*

(2) *$m(z)$ has no relative weak maximum points and no relative weak minimum points other than zeros.*

Note. For real valued functions on R the absence of relative maximum and minimum points implies strong interiority.

(3) Maximum Modulus Theorem. *If $f(z)$ is analytic in a bounded open set G and on $\mathrm{Fr}(G)$ and $|f(z)| \leqq M$ on $\mathrm{Fr}(G)$, then $|f(z)| < M$ for $z \in G$.*

For otherwise by (1) $m(z)$ exceeds M for some $z \in G$. Then $m(a) = \max_{z \in \bar{G}} m(z)$ for some $a \in G$, which is impossible by (1).

(4) *If $f(z)$ is analytic in a bounded open set G complementary to a level curve L: $|f(z)| = k$ and on $\mathrm{Fr}(G)$ where $\mathrm{Fr}(G) \subset L$, $f(z)$ has at least one zero in G.*

For by (3), $|f(z)| < k$ on G. Thus $m(a) = \min_{z \in \bar{G}} m(z)$ for some $a \in G$; and by (1) $m(a)$ must be 0.

(5) *Every (non-constant) polynomial has at least one zero.*

For let $w_0 \neq 0$ be a value of a polynomial $P(z)$. The open set G defined by $|P(z)| < |w_0|$ is bounded, since $P(z) \to \infty$ as $z \to \infty$, and $\mathrm{Fr}(G)$ is contained in the level curve $|P(z)| = |w_0|$. Thus the result follows from (4).

(6) *Let R be any bounded region with connected boundary B and let $f(z)$ be analytic on $R + B$. Then either $m(R) \subset m(B)$ or else f has at least one zero in R.*

For the set $m(B)$ is a continuum and hence is an interval $\alpha \leqq x \leqq \beta$ of the non-negative real axis; and if it does not contain $m(R)$, we have $\alpha > m(z_0) > 0$ for some $z_0 \in R$ by (3). Thus $m(a) = \min_{z \in R} m(z)$ for some $a \in R$ and $m(a) = 0$ by (1).

Remark. It seems likely that while the strong interiority of $m(z)$ is an immediate consequence of that of $f(z)$ it might well be much easier to establish by an elementary approach than is the strong interiority of $f(z)$. For example, the property obtained in the conclusion of (3.1) that every linear interval starting from w_0 intersect $f(S)$ again yields strong interiority for $m(z)$ but not for $f(z)$.

5. BOUNDARY PERMUTATION AND REGION PRESERVATION. In this section A and B_0 will designate locally connected generalized continua, i.e., locally compact, connected, locally connected separable, metric spaces.

(5.1) Theorem. *If $f(A) = B \subset B_0$ is strongly interior, for any conditionally compact open set U in A, we have $\mathrm{Fr}[f(U)] \subset f[\mathrm{Fr}(U)]$.*

As U is conditionally compact, continuity of f gives $\overline{f(U)} = f(\bar{U})$ which in turn gives $\overline{f(U)} - f(U) \subset f(\bar{U} - U)$; and since $f(U)$ is open, this latter is the same as our conclusion.

(5.2) Theorem. *Let $f(A) = B \subset B_0$ be strongly interior. Then for any region R in B_0, any non-empty conditionally compact component of $f^{-1}(R)$ maps onto R under f.*

Proof. Let Q be a conditionally compact component of $f^{-1}(R)$. Then Q, being a component of an open set of A, is open. Hence $f(Q)$ is open in B_0 and therefore also in R. We show also that $f(Q)$ is closed in R. Firstly, no points of

$\mathrm{Fr}(Q)$ are in $f^{-1}(R)$. For suppose x belongs to $\mathrm{Fr}(Q)$ and $f^{-1}(R)$. Then $Q + x$ is a connected subset of $f^{-1}(R)$ containing the component Q as a proper subset. This is impossible and $\mathrm{Fr}(Q)$ and $f^{-1}(R)$ are disjoint. Since $f[\mathrm{Fr}(Q)] \supset \mathrm{Fr}[f(Q)]$ and $f[\mathrm{Fr}(Q)]$ does not intersect R, $f(Q)$ is closed in R. Then $f(Q) = R$, being both open and closed in R.

Corollary. *Let $w = f(z)$ be analytic in a bounded region R and on* $\mathrm{Fr}(R)$. *Any value w_0 taken by $f(z)$ on $\bar{R}$ lies either in $f[\mathrm{Fr}(R)] = F$ or in some bounded component of $W - F$.*

If f takes in R one value w_0 in a component Q of $W - F$, then it takes all values in Q.

6. EXPANSIVE MAPPINGS. As in the preceding section A and B_0 will denote locally connected generalized continua. A mapping $f(A) = B \subset B_0$ is said to be *expansive* on A provided A is the sum of a strictly monotone increasing sequence $[R_n]$ of conditionally compact regions with $\bar{R}_n \subset R_{n+1}$ such that if $F_n = f[\mathrm{Fr}(R_n)]$, then any compact set K in B_0 intersects at most a finite number of the sets F_n.

Remark. An entire analytic mapping $w = f(z)$ is expansive provided that for some sequence of circles C_n:

$$|z_n| = r_n \text{ with } r_n \to \infty, \text{ we have } \lim_{n\to\infty} [\min_{|z|=r_n} |f(z)|] = \infty.$$

(6.1) Theorem. *If $f(A) = B \subset B_0$ is strongly interior and expansive, we have $B = B_0$. In other words, every point of B_0 is the image of some point of A.*

Proof. Let $y \in B_0$; we must show that $y \in f(A)$. Let $R_1, R_2, \ldots$ be a strictly monotone increasing sequence of regions in A as guaranteed by the expansibility of f, and let $F_n = f[\mathrm{Fr}(R_n)]$. If $y \in F_1$, then $y \in f(A)$. If $y \notin F_1$ then join y to F_1 by a simple arc yz, where $z \in F_1$. Let x be a point of $\mathrm{Fr}(R_1)$ with $f(x) = z$. Since f is expansive we may find an integer k, $k > 1$, such that for $n \geq k$, then $F_n \cdot (yz) = 0$. By strict monotoneity of $[R_n]$, $x \in R_k$. Also $z \notin f[\mathrm{Fr}(R_k)] = F_k$. Let Q be the component of $B_0 - F_k$ which contains z. Let U be the component of $f^{-1}(Q)$ which contains x. Since U is a connected subset of $A - f^{-1}(F_k) \subset A - \mathrm{Fr}(R_k)$ which intersects R_k, we must have $U \subset R_k$. Then U is conditionally compact. By (5.2) we then have $f(U) = Q$. But $y \in Q$, since yz is a connected set not intersecting F_k. Hence there is a point of R_k which maps onto y, and the theorem is proved.

(6.11) Corollary. *An expansive, entire analytic function takes on all finite complex values.*

(6.12) Corollary. *Every polynomial has at least one zero.*

Definition. Let $f(A) = B \subset B_0$ be continuous. We say that $\lim_{x\to\infty} f(x) = \infty$ if for every sequence $x_1, x_2, \ldots$ in A without a limit point it is true that $f(x_1)$, $f(x_2), \ldots$ contains no subsequence which converges to a limit in B_0.

(6.2) Theorem. *If $f(A) = B_0$ is strongly interior and expansive, then either $\lim_{x\to\infty} f(x) = \infty$ or else every $f^{-1}(z)$, $z \in B_0$, is infinite. Further if $\lim_{x\to\infty} f(x) = \infty$ then $f^{-1}(z)$ is compact for every $z \in B_0$; if $\lim_{x\to\infty} f(x) \neq \infty$, $f^{-1}(z)$ is non-compact for every $z \in B_0$.*

Proof. We assume that $\lim_{x\to\infty} f(x) \neq \infty$ and show that each $f^{-1}(z)$, $z \in B_0$, is non-compact. By assumption, there is an infinite sequence $[x_i]$ of distinct points in A without a limit point such that $[f(x_i)]$ converges to a point y of B_0. Since f is expansive, there exists a strictly monotone increasing sequence $[R_n]$ of conditionally compact regions with union A such that no compact subset of B_0 intersects infinitely many F_n, where $F_n = f[\mathrm{Fr}(R_n)]$. Since the sequence $[x_i]$ has no limit point in A, it follows that no region R_n contains infinitely many points of $[x_i]$. Hence the sequences $[R_n]$ and $[x_n]$ may be considered adjusted so that $x_1 \in R_1$, $x_n \in R_n - \bar{R}_{n-1}$ for $n > 1$.

Choose any $z \varepsilon B$. Let D be a continuum in B_0 containing y and z. Let C be a conditionally compact region in B_0 containing y, and let $K = \overline{C} + D$. Then K is a continuum in B_0 containing y and z and with y an interior point of K. Since the sequence $[f(x_i)]$ converges to y, then $f(x_n) \varepsilon K$ for $n \geq k$. Moreover we can assume k large enough so that $K \cdot F_n = 0$ for $n \geq k$. Let Q_n be the component of $B_0 - F_n - F_{n-1}$ containing K for $n > k$. Let W_n be the component of $A - f^{-1}(F_n + F_{n-1})$ which contains x_n. Since $x_n \varepsilon R_n - \overline{R}_{n-1}$ and $f^{-1}(F_n + F_{n-1}) \supset \mathrm{Fr}(R_n) + \mathrm{Fr}(R_{n-1})$, then $W_n \subset R_n - \overline{R}_{n-1}$ for $n > k$. Since W_n is a conditionally compact component of $f^{-1}(Q_n)$, then by (5.2) we have $f(W_n) = Q_n$. Then, for $n > k$ there is a point q_n of $R_n - \overline{R}_{n-1}$ with $f(q_n) = z \varepsilon Q_n$. Hence $f^{-1}(z)$ is non-compact and the theorem is proved.

Examples of expansive functions.

(1) Every polynomial is expansive. Any rational function is expansive on the set $S - P$ where S is the complex sphere and P is the set of all poles of the function.

(2) The exponential e^z is not an expansive function. To see this, consider a conditionally compact region R containing the origin. Then $\mathrm{Fr}(R)$ must intersect the y-axis, whose image is the unit circle. Hence $f[\mathrm{Fr}(R)]$ intersects the unit circle, which is compact. This is easily seen to contradict expansibility.

(3) Any non-constant entire function of order less than 1/2 is expansive. For by the well known theorem of Wiman [4] any such function has the property quoted in the remark at the beginning of this section.

The entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is of order ρ if and only if

$$\varlimsup_{n\to\infty} \frac{\log \frac{1}{|a_n|}}{n \log n} = \frac{1}{\rho}.$$

As a consequence, $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{\alpha}}$ is of order $\frac{1}{\alpha}$. (See Titchmarsh [3], pages 253-255).

Consider the function $f(z) = \prod_{n=1}^{\infty} (1 - \frac{z}{2^n})$. This function is an entire function. The order of convergence of the zeros of $f(z)$ (i.e., the g.l.b. of all ρ for which $\sum_{n=1}^{\infty} (\frac{1}{2^n})^{\rho}$ is convergent) is zero, and the order of an infinite product is equal to the order of convergence of its zeros; thus we see that $f(z)$ is of order zero. By our theorem, $f(z)$ is expansive. This fact can be verified directly by observing that for each $r > 0$, $\min_{|z|=r} |f(z)| = |f(r)|$.

If we define $r_k = \frac{2^k + 2^{k+1}}{2} = 3.2^{k-1}$, it may be verified that $f(r_{k+1}) =$ $= (1 - 3.2^{k-1}) \cdot f(r_k) \neq 0$. Thus $\lim_{k\to\infty} |f(r_k)| = \infty$. If we define R_k to be the set of all z such that $|z| \leqq r_k$, we see that $f[\mathrm{Fr}(R_k)]$ can intersect a given compact set for at most a finite number of values of k.

(6.3) Theorem. *A mapping $f(A) = B \subset B_0$ is expansive if and only if each component of the inverse of a continuum in B_0 is compact.*

Proof. (Necessity) Assuming the map f expansive, we show that each component of the inverse of any continuum H in B_0 is compact. Suppose on the contrary that H is a continuum in B_0 having Q as a non-compact component of its inverse $f^{-1}(H)$. Then Q is closed but not compact. Hence for no finite n is Q contained in R_n. It follows that Q, being connected, must intersect the boundary of infinitely many R_n. It follows that for infinitely many n, $f(Q) \cdot f[\mathrm{Fr}(R_n)] \neq 0$. Since $f(Q) \subset H$, H is then a compact set intersecting infinitely many F_n. This contradicts the expansitivity of f.

(Sufficiency) Suppose $[W_n]$ and $[Q_n]$ are strictly monotone increasing sequences of conditionally compact regions whose sums are A and B_0 respectively, Suppose these selected so that $f(W_1) \cdot Q_1 \neq 0$. Define $E_1 = \bar{Q}_1 + f(\bar{W}_1)$. Then E_1 is a continuum since both $\bar{Q}_1$ and $f(\bar{W}_1)$ are continua and $\bar{Q}_1 \cdot f(\bar{W}_1) \neq 0$. Let K_1 be the component of $f^{-1}(E_1)$ containing W_1. By hypothesis K_1 is compact and hence a continuum. It is possible to find a conditionally compact neighborhood R_1 of K_1 whose boundary does not intersect $f^{-1}(E_1)$. Then $K_1 \subset R_1$ and if $F_1 = f[\mathrm{Fr}(R_1)]$, it is true that $F_1 \cdot E_1 = 0$.

Let $E_2 = \bar{Q}_2 + f(\bar{R}_1) + f(\bar{W}_2)$. Then E_2 is a continuum, being the sum of three intersecting continua. Let K_2 be the component of $f^{-1}(E_2)$ containing $\bar{R}_1$. As before, we can find a conditionally compact region R_2 containing K_2 and such that $F_2 \cdot E_2 = 0$, where $F_2 = f[F(R_2)]$. Continuing the process we define

$$E_n = \bar{Q}_n + f(\bar{W}_n) + f(R_{n-1}).$$

This is a continuum, being the union of three intersecting continua. Define K_n to be the component of $f^{-1}(E_n)$ containing $\bar{R}_{n-1}$. Let R_n be a conditionally compact region in A containing K_n and such that $f[\mathrm{Fr}(R_n)] \cdot E_n = 0$. We obtain a strictly monotone increasing sequence $[R_i]$ of conditionally compact regions whose sum is A. Let $F_n = f[\mathrm{Fr}(R_n)]$. If C is a compact subset of B_0, there exists an integer k so that $C \subset Q_k$. But then $C \subset E_n$ for $n \geq k$ and hence $C \cdot F_n = 0$ for $n \geq k$. This proves that f is expansive.

(6.31) Corollary. *A mapping f of A onto B_0 is expansive if and only if non-compactness is invariant under f for closed generalized continua of A.*

Examples.

(1) The mapping $f(z) = e^z$ takes the non-compact y-axis into the compact unit circle, and hence is non-expansive.

(2) The function $f(z) = \cos z$ maps the closed non-compact x-axis onto the compact interval $-1 \leqq u \leqq 1$. Hence f is the non-expansive.

(3) By a topological ray we mean the set obtained by removing the point ∞ from an arc joining the origin to ∞. If w_1 is the limit of $f(z)$ as z approaches ∞ along some ray R, then w_1 is an *asymptotic value* of $f(z)$. A necessary condition for expansibility of a function is the absence of asymptotic values.

In connection with (3) we have the following characterization of expansibility of mappings.

(6.4) Theorem. *If A is a locally connected generalized continuum and B_0 is locally compact separable and metric, a mapping $f(A) = B_0$ is expansive if and only if on each closed topological ray in A there exists a sequence (x_n) of points such that $f(x_n) \to \infty$ (i.e., no subsequence of $[f(x_n)]$ converges to a point).*

For let f be expansive. Then if $R_1, R_2, \ldots$ is a strictly monotone increasing sequence of conditionally compact regions with union A such that no compact set in B_0 intersects infinitely many of the sets $f[\mathrm{Fr}(R_n)]$ and r is any closed topological ray in A, there exists an $x_n \in r \cdot \mathrm{Fr}(R_n)$ for n sufficiently large. Clearly $f(x_n) \to \infty$, since no compact set in B_0 contains infinitely many of the points $[f(x_n)]$.

On the other hand suppose f is not expansive. Then by (6.31) there exists a non-compact closed generalized continuum N in A such that $f(N) = M$ is compact. Using the local connectedness of A and the local compactness of A and B_0 we can readily construct a closed locally connected generalized continuum K in A containing N such that $f(K) = H$ is a continuum. For let $N = \sum_1^{\infty} N_i$ where each N_i is compact and $N_1 \subset N_2 \subset \ldots$ and let $\varepsilon > 0$ be chosen so that $\overline{V_{2\varepsilon}(M)}$ is compact. For each i, N_i is interior rel. A to a locally connected continuum K_i satisfying

$$K_i \subset V_{1/i}(N_i),\ f(K_i) \subset V_{\varepsilon/i}(M).$$

Then if $K = \Sigma K_i$, $H = \Sigma f(K_i) = f(K)$, it is readily seen that K is a closed locally connected generalized continuum and that H is closed and hence is a continuum. But now K contains a closed topological ray r, since $K \supset N$ and thus is non-compact; and since $f(r) \subset H$ and H is compact, clearly for no sequence (x_n) in r can we have $f(x_n) \to \infty$.

7. QUASI-INTERIOR MAPPINGS. A mapping $f(A) = B \subset B_0$ is said to be *quasi-interior* provided that for any $y \varepsilon B$ and any open set U in A containing a compact component of $f^{-1}(y)$, y is interior relative to B_0 to $f(U)$.

Notes.

1. Every strongly interior mapping is quasi-interior.
2. Every light quasi-interior mapping is strongly interior.

(7.1) Theorem. *On a locally connected generalized continuum A, a mapping $f(A) = B \subset B_0$, where B_0 is locally connected, is quasi-interior if and only if for each region R in B_0, each conditionally compact component of $f^{-1}(R)$ maps onto R under f.*

Proof. Necessity. Let Q be a non-empty conditionally compact component of $f^{-1}(R)$. Let $y \varepsilon f(Q)$. Then Q contains a compact component of $f^{-1}(y)$. For if $x \varepsilon Q \cdot f^{-1}(y)$, the component of $f^{-1}(y)$ containing x lies in Q, is a closed set, and thus is compact since Q is conditionally compact. Accordingly, y is interior to $f(Q)$. Then if $f(Q) \neq R$ there is a point $z \varepsilon R - f(Q)$ which is a limit point of $f(Q)$. Let $[z_i]$ converge to z, where $z_i \varepsilon f(Q)$, let $x_n \varepsilon Q \cdot f^{-1}(z_n)$. Since Q is conditionally compact, we may suppose that $[x_n]$ converges to a point $x \varepsilon \bar{Q}$. Then $f(x) = \lim f(x_i) = \lim z_i = z$, and hence $x \varepsilon f^{-1}(R)$. Then $x \varepsilon Q$, since Q is a component of $f^{-1}(R)$. This implies that $z \varepsilon f(Q)$, which is a contradiction.

Sufficiency. We must show f quasi-interior. Let $y \varepsilon B$ and let X be a compact component of $f^{-1}(y)$. Let U be an open set containing X. Let V be an open set containing X such that $V \subset U$, $\bar{V}$ is compact, and $f^{-1}(y) \cdot \mathrm{Fr}(V) = 0$. Then $y \not\varepsilon f[\mathrm{Fr}(V)]$. Choose a region R in B_0 so that $y \varepsilon R$ and $R \cdot f[\mathrm{Fr}(V)] = 0$, which is possible since $f[\mathrm{Fr}(V)]$ is a compact set not containing y. Since $\bar{V}$ is compact and since $f^{-1}(R) \cdot \mathrm{Fr}(V) = 0$, there is a conditionally compact component Q of $f^{-1}(R)$ contained in V. By hypothesis, $f(Q) = R$ and hence y is interior to $f(U)$.

Note: *Neither local connectedness nor local compactness is invariant under all quasi-interior mappings*. For we may map the interval $0 < t \leqq 1$ homeomorphically into the graph C in the xy-plane of $y = \sin \pi/x$, $0 < x \leqq 1$. Map the interval $1 \leqq t \leqq 2$ homeomorphically into an arc in the xy-plane which only intersects C at $(1,0)$, 1 mapping into $(1,0)$ and 2 mapping into $(0,0)$. Map the entire interval $2 < t < 3$ into $(0,0)$. We have now defined a quasi-interior map of the locally connected and locally compact interval $0 < t < 3$ into a set which is neither locally compact nor locally connected. (The map is quasi-interior since the inverse of $(0,0)$ has no compact component.)

We next prove a theorem on sequences of quasi-interior mappings analogous to the Weierstrass double series theorem. For a closely related result see [5]. Actually since quasi-interiority is equivalent on locally connected continua to quasi-monotoneity (see below), this theorem includes the main reresult of [5] as a particular case.

(7.2) Theorem. *If the sequence of quasi-interior mappings $f_n(A) = B_n \subset B_o$ converges to the mapping $f(A) = B \subset B_o$, the convergence being uniform on each compact set in A, then f is quasi-interior.*

Proof. Let $y \in B$ and let X be a non-empty compact component of $f^{-1}(y)$. Let W be an open set containing X. Choose an open set U containing X such that $\bar{U}$ is compact, $\bar{U} \subset W$, and such that $f^{-1}(y) \cdot \mathrm{Fr}(U) = 0$. We show that y is interior to $f(\bar{U})$.

Define e so that $3e = \rho(y,F)$, where $F = f[\mathrm{Fr}(U)]$. Then $e > 0$ and we may choose $0 < \varepsilon < e$ such that $V_\varepsilon(y)$ is contained in some component R of $V_e(y)$. Since the convergence is uniform on $\bar{U}$, there exists an integer N such that for $n \geq N$ we have $\rho[f_n(x), f(x)] < \varepsilon$ for all $x \in \bar{U}$. Then for $n > N$ we have $f_n(X) \subset V_\varepsilon(y) \subset R$ and $F_n \equiv f_n[\mathrm{Fr}(U)] \subset V_\varepsilon(F)$. Then $R \cdot F_n = 0$, and R is contained in the component R_n of $B_o - F_n$ which contains y. Then if U_n is the component of $f^{-1}(R_n)$ containing X, we have $U_n \subset U$. Then $f_n(U_n) = R_n \supset R \supset V_\varepsilon(y)$ and hence $f_n(U) \supset R$ for n sufficiently large. Since $\bar{U}$ is compact, we obtain $f(\bar{U}) \supset R$. This completes the proof.

8. COMPACT MAPPINGS. A mapping $f(A) = B$ is said to be *compact* provided the inverse of every compact set in B is compact. Also f is said to be *monotone* provided that for each $y \in B$, $f^{-1}(y)$ is a continuum; and f is said to be *strongly monotone* provided the inverse of every continuum in B is a continuum. All spaces used in this section are supposed separable and metric.

Notes.

1. Compactness of a mapping is equivalent to invariance of non-conditional compactness.

2. Compactness of a mapping is equivalent to invariance of non-compactness for closed sets.

3. If A is compact, every mapping on A is compact.

(8.1) Theorem. *Suppose $f(A) = B$ is compact. Then for any $y \in B$ and any open set U containing $f^{-1}(y)$, y is interior to $f(U)$.*

Proof. Suppose there exists an open set U containing $f^{-1}(y)$ such that y is not interior to $f(U)$. Then there exists a sequence $[y_i]$ in $B - f(U)$ converging to y. Consider the compact set $K = y + \sum_{i=1}^{\infty} y_i$. By compactness of f, $f^{-1}(K)$ is compact. Let $x_n \in f^{-1}(y_n)$. Then $x_n \notin U$. Since $f^{-1}(K)$ is compact, there is a convergent subsequence $[x_{n_i}]$ of A with $x_{n_i} \to x \in A$. Then $f(x_{n_i}) \to f(x)$. But $f(x_{n_i}) = y_{n_i} \to y$ and hence $x \in f^{-1}(y)$. Since $x \notin U$ and $f^{-1}(y) \subset U$, we have a contradiction.

(8.11) Corollary. *If A is compact, every monotone mapping on A is quasi-interior.*

(8.12) Corollary. *Local compactness and local connectedness are invariant under compact mappings.*

(8.2) Theorem. *A mapping $f(A) = B$ is compact if and only if it is closed and point-inverses are compact.**

*Thus our definition of compact mapping is equivalent to Vainstein's. See [6,7]. The term is due to Vainstein. Out results were obtained independently of his using the term "strong continuity." Compare also with R. Manning [8].

Proof. Suppose f is compact. It follows immediately that point-inverses are compact. Suppose now that C is closed in A while $f(C)$ is not closed in B. Then there exists a point $y \in B - f(C)$ and a sequence $y_i \to y$ where $y_i \in f(C)$. Then $K = y + \sum_{i=1}^{\infty} y_i$ is a compact set and hence $f^{-1}(K)$ is compact. Let $x_n \in C \cdot f^{-1}(y_i)$. Then there exists a point $x \in f^{-1}(K)$ and a subsequence $[x_{n_i}]$ such that $x_{n_i} \to x$. Then $f(x_{n_i}) = y_{n_i} \to f(x)$ and $y = f(x)$. Since C is closed, $x \in C$ and $f(x) = y \in f(C)$. This is a contradiction.

Suppose f is closed and point-inverses are compact. Suppose C is a compact set in B such that $f^{-1}(C)$ is not compact. Let $[x_i]$ be an infinite sequence of distinct points of $f^{-1}(C)$ without a limit point. Since point-inverses are compact, we may suppose that $f(x_i) \neq f(x_j)$ for $i \neq j$. Then $K = \sum_{i=1}^{\infty} x_i$ is a closed set in $f^{-1}(C)$; and $f(K)$ is a countably infinite compact subset of C. There exists a subsequence of $[f(x_i)]$, which we suppose the same as the original sequence, which converges to a point $y \in C$ distinct from any $f(x_i)$. Then $\sum_{i=1}^{\infty} x_i$ is a closed subset of A which maps onto the set $\sum_{i=1}^{\infty} f(x_i)$, which is not closed. This is a contradiction and $f^{-1}(C)$ must be compact.

(8.3) Theorem. *If $f(A) = B$ is compact and monotone, then the inverse of every connected set in B is connected.*

Proof. Let C be a connected set in B and suppose that $f^{-1}(C) = S_1 + S_2$ is a separation. Since f is monotone, S_1 and S_2 are inverse sets. We have $S_1 \cdot \bar{S}_2 = 0$. Since S_1 is an inverse set, this implies that $f(S_1) \cdot f(\bar{S}_2) = f(S_1 \cdot \bar{S}_2) = 0$. Since f is closed, we have $f(\bar{S}_1) = \overline{f(S_1)}$; and hence $f(S_1) \cdot f(S_2) = 0$. Similarly $f(S_2) \cdot f(S_1) = 0$. Therefore $C = f(S_1) + f(S_2)$ is a separation of C and we have a contradiction.

(8.31) Corollary. *A compact monotone mapping is strongly monotone, i.e., the inverse of a continuum is a continuum.*

(8.4) Theorem. *If A is locally compact and $f(A) = B$ is monotone, then a necessary and sufficient condition that f be compact is that it be quasi-interior.*

Proof. The necessity follows immediately from (8.1). Suppose now that f is quasi-interior and C is a compact subset of B. Let $[x_i]$ be a sequence in $f^{-1}(C)$; we wish to show that $[x_i]$ has a convergent subsequence. We may suppose that $[f(x_i)]$ converges to a point $y \in C$ without loss of generality. Let U be a conditionally compact open set containing $f^{-1}(y)$. Almost all $f^{-1}f(x_i)$ intersect U by the quasi-interiority of f. We show that almost all $f^{-1}f(x_i)$ are contained in U. If this were not true, the connectedness of each $f^{-1}f(x_i)$ would enable us to find an infinite sequence of points $z_{n_i} \in f^{-1}f(x_{n_i})$ such that $z_{n_i} \to z$ where $z \in \mathrm{Fr}(U)$. This implies that $z \in f^{-1}(y)$ which is a contradiction. Therefore almost all $f^{-1}f(x_i)$ are contained in U. Then almost all $[x_i]$ are contained in $\bar{U}$, which is compact. It follows that $f^{-1}(C)$ is compact.

(8.5) Theorem. *If A and B are locally connected generalized continua then for a monotone mapping $f(A) = B$ the properties of compactness, strong monoteneity, and quasi-interiority are equivalent.*

Proof. By the preceding theorem, compactness of f is equivalent to quasi-interiority. We have also shown previously that compactness of f implies strong monoteneity. Suppose now that f is strongly monotone and let K be a compact subset of B. There exists a continuum C in B containing K. Then $f^{-1}(C)$ is a continuum containing the closed set $f^{-1}(K)$. Therefore $f^{-1}(K)$ is compact.

9. COMPOSITION OF MAPPINGS. A type of mapping is understood to have the the "group property" provided the composition of two mappings of this type is also of this type. For example, if $f_1(A) = B$ and $f_2(B) = C \subset C_0$ are strongly interior, so also is $f_2f_1(A) = C \subset C_0$ as is well known. Also, monotone mappings on compact spaces have the group property. The following facts are readily verified.

1. Compact mappings have the group property.
2. Strongly monotone mappings have the group property.
3. Monotone mappings do not have the group property (on non-compact spaces).
4. Quasi-interior mappings do not have the group property.

(9.1) Theorem. *If $f_1(A) = B$ is compact and monotone and $f_2(B) = C$ is quasi-interior, then $f = f_2f_1$ is quasi-interior.*

Proof. Let X be a compact component of $f^{-1}(y)$, $y \in B$, and let U be any open set containing X. Since f_1 is strongly monotone, $f_1(X)$ is a component of $f_2^{-1}(y)$. Since f_1 is quasi-interior, $f_1(X)$ is in the interior of $f_1(U)$. Since f_2 is quasi-interior, we have that y is an interior point of $f_2f_1(U) = f(U)$.

(9.2) Theorem. *If f_1 is quasi-interior and f_2 is interior, then $f = f_2f_1$ is quasi-interior.*

Proof. Let $f_1(A) = M$ and $f_2(M) = B$. Consider any $y \in B$, let X be a compact component of $f^{-1}(y)$ and let U be any open set containing X. Then $f_1(X)$ is a continuum. Let $x \in f_1(X)$; then any component of $f_1^{-1}(x)$ which intersects X has to be contained in X and thus in U, so x is interior to $f_1(U)$. Hence $f_1(X)$ is contained in the interior of $f_1(U)$. Since f_2 is interior, $y = f_2f_1(X)$ is interior to $f_2f_1(U)$. Therefore f is quasi-interior.

10. FACTORIZATION. By a factorization of a mapping $f(A) = B$ is meant a decomposition of f into two mappings $f_1(A) = M$ and $f_2(M) = B$ such that $f(x) = f_2f_1(x)$ for each $x \in A$. The space M is called the *middle space* (Rado) and it, like A and B, is supposed separable and metric.

(10.1) Theorem. *If $f(A) = B$ is compact, then however it be factored into the form $f = f_2f_1$ where the transformations f_1 and f_2 are continuous, f_1 and f_2 are compact.*

Proof. Let $M = f_1(A)$. If $Y \subset B$ is compact, $f^{-1}(Y)$ is compact since f is compact. Then $f_1f^{-1}(Y) = f_2^{-1}(Y)$ is compact by continuity of f_1.

For any $Z \subset M$, $f_1^{-1}(Z) \subset f^{-1}f_2(Z)$. If Z is compact, so also is $f_2(Z)$. Since f is compact, $f^{-1}f_2(Z)$ is then compact. Then $f_1^{-1}(Z)$, being a closed subset of a compact set, is compact. Hence f_1 is compact.

Note. The preceding theorem shows that however a compact mapping f be factored continuously into $f = f_2f_1$, both f_1 and f_2 are compact. It is also true if under f point inverses are compact then however it be factored continuously into $f = f_2f_1$, under both f_1 and f_2 point inverses are compact. However, if f is closed, it does not follow that both factors f_1 and f_2 of f are necessarily closed. On the other hand, since $f_2(Y) = ff_1^{-1}(Y)$, it follows that f_2 would always be closed.

(10.2) Theorem.* *Any compact mapping admits a factorization $f = f_2f_1$ where f_1 is compact and monotone and where f_2 is compact and light.*

Proof. We first construct the middle space M. A point of M is defined to be a component of $f^{-1}(y)$ for some $y \in B$. A subset U of M is defined to be open

*This theorem is stated without proof in Vainstein [6]. It was discovered independently by the author and the proof is included for the sake of completeness; it differs some from the proof in the well-known case of compact spaces as proven in 1934 by Eilenberg and the author.

in M if and only if the union of the elements of U is open in A. We proceed to verify the axioms for a regular perfectly separable topological space. The union and intersection axioms are obvious. We prove a lemma before verifying the separation axiom.

Lemma. *Let $f(A) = B$ be compact. If U is any open set in A and U_0 is the union of all components of sets $f^{-1}(y)$, $y \varepsilon B$, contained in U, then U_0 is open in A.*

Proof. Suppose the lemma is false. There exists a point $p \varepsilon U_0$ and a sequence $[p_i]$ in $A - U_0$ such that $p \varepsilon \liminf C_i$, where C_i is the component of $f^{-1}f(p_i)$ containing p_i. Let $K = f(p) + \sum_{i=1}^{\infty} f(p_i)$. Since K is compact, $f^{-1}(K)$ is also compact. Accordingly $\limsup C_i$ is connected. Since f is continuous, $\limsup C_i \subset f^{-1}f(p)$. Since $C_i \not\subset U$ and $f^{-1}(K)$ is compact, $\limsup C_i \not\subset U$. Consequently the component of $f^{-1}f(p)$ containing p is not contained in U. This is a contradiction, since $p \varepsilon U_0$.

Using the lemma, we may now establish the separation axiom. Let x and y be distinct points of M. Let X be the closed subset of A corresponding to x. Then $A - X$ is open, and hence the elements z of M such that Z is contained in $A - X$ forms an open subset of A. The element y does not belong to the open set of M determined by $A - X$.

We proceed to verify regularity of M. Let $p \varepsilon M$ and let U' be an open set in M containing p. Let P and U be the sets of A corresponding to p and U' respectively. There exists an open set V in A such that $P \subset V \subset \bar{V} \subset U$. Let V_0 be the set of all points in M whose correspondents are contained in V. By the lemma, this is an open subset of M. We show that $\bar{V}_0 \subset U'$. Let q be a point of $M - U'$. Then Q, the set of A corresponding to q, is contained in $A - U$. There exists an open set W containing Q and such that $V \cdot W = 0$. If W_0 is the subset of M consisting of all points of M whose correspondents in A are contained in W, then W_0 is an open subset of M containing q such that $V_0 \cdot W_0 = 0$.

To show M perfectly separable, consider a fundamental sequence (basis) $[R_n]$ of open sets in A. For any finite set $n_1, \ldots n_k$ of positive integers, let $R(n_1, \ldots, n_k) = R_{n_1} + R_{n_2} + \ldots + R_{n_k}$. Let $R_0(n_1, \ldots, n_k)$ be the subset of M consisting of all points of M whose correspondents are contained in $R(n_1, \ldots, n_k)$. Then each $R_0(n_1, \ldots, n_k)$ is an open set. Let $p \varepsilon M$ and let U' be an open set in M containing p. Let P, U be the subsets of A corresponding to p, U' respectively. Since P is compact, there exists integers $n_1, \ldots, n_k$ such that $P \subset R(n_1, \ldots, n_k) \subset U$. Then $p \varepsilon R_0 (n_1, \ldots, n_k) \subset U'$. Thus the sets R_0 form a basis in M.

We now define the factors f_1 and f_2. For each $x \varepsilon A$, let $f_1(x)$ be the element of M whose correspondent contains x. For each $p \varepsilon M$, let $f_2(p) = ff_1^{-1}(p)$. By the definition of open sets in M, f_1 is continuous. We show that f_1 is closed. Let K be a closed set in A. Then $M - f_1(K)$ is the set of all elements y of M such that $f_1^{-1}(y) \subset A - K$. Therefore $M - f_1(K)$ is open in M. Hence $f_1(K)$ is closed in M and f_1 is closed. The map f_2 is continuous; for if Y is any closed set in B then $f_2^{-1}(Y) = f_1 f^{-1}(Y)$ is closed in M. The map f_1 is clearly monotone. Both f_1 and f_2 are compact by (10.1). We show finally that f_2 is light. Suppose f_2 is not light. There exists then a non-degenerate continuum K in M such that $f_2(K)$ is a point y of B. Then $f_1^{-1}(K)$ is a continuum contained in $f^{-1}(y)$. Since $f_1^{-1}(K)$ is contained in a component of $f^{-1}(y)$, we have that $f_1 f_1^{-1}(K) = K$ is a single point, which is a contradiction.

Definition. Two mappings $f(A) = B$ and $g(A') = B'$ (all spaces topological) are *topologically equivalent* if there exist homeomorphisms $h(A) = A'$ and

$k(B) = B'$ such that for $x \varepsilon A$, $f(x) = k^{-1}gh(x)$. Two factorizations $f = f_2 f_1$, $f = g_2 g_1$ of a mapping $f(A) = B$ are *strictly topologically equivalent* provided there exists a homeomorphism $h(M) = N$ from the middle space M of the first to the middle space N of the second such that $hf_1(x) \equiv g_1(x)$, $x \varepsilon A$, and $f_2(y) \equiv g_2 h(y)$, $y \varepsilon M$.

(10.3) Theorem. *Any two monotone-light factorizations of a compact mapping are strictly topologically equivalent.*

Proof. Let $f(A) = B$ be compact and let $f = f_2 f_1$, where $f_1(A) = M$ and $f_2(M) = B$, and $f = g_2 g_1$, where $g_1(A) = N$ and $g_2(N) = B$ be monotone-light factorizations of f.

For each $x \varepsilon M$ let $h(x) = g_1 f_1^{-1}(x)$ so that for each $y \varepsilon N$, $h^{-1}(y) = f_1 g^{-1}(y)$. Since for any $x \varepsilon M$, $f_1^{-1}(x)$ is a component of $f^{-1}f_2(x)$ and similarly for $y \varepsilon N$, $g_1^{-1}(y)$ is a component of $f^{-1}g_2(y)$, it follows that both h and h^{-1} are single valued. Further, if K is any closed set in N, $g_1^{-1}(K)$ is closed by the continuity of g_1 and $f_1 g_1^{-1}(K) = h^{-1}(K)$ is then closed since f_1 is closed. Similarly for any closed set H in M, $h(H) = g_1 f_1^{-1}(H)$ is closed since f_1 is continuous and g_1 is closed. Thus h is a homeomorphism.

(10.4) Theorem. *A compact mapping f is quasi-interior if and only if it factors into the form $f_2 f_1$ where f_1 is monotone and f_2 is light and interior.*

Proof. Suppose that f is quasi-interior. Since f is compact, it admits a monotone-light factorization $f = f_2 f_1$. We must prove that f_2 is interior.

Let x be a point in the domain of f_2 and let U be a neighborhood of x. Since f_1 is compact and monotone, $X = f_1^{-1}(x)$ is a continuum. The set X is a component of $f^{-1}f_2(x)$. Let $V = f_1^{-1}(U)$. Then V is an open set containing X; and therefore, since f is quasi-interior, $f_2(x)$ is interior to $f(V)$. However, $f(V) = f_2(U)$. This proves that f_2 is interior.

Now consider a compact mapping which admits a monotone-light factorization $f = f_2 f_1$ where f_2 is also interior. Since f_1 is monotone and compact, f_1 is quasi-interior. Since we are assuming f_2 interior, it follows from (9.2) that $f = f_2 f_1$ is quasi-interior.

Definition. (Wallace) A mapping $f(A) = B$, where A is a locally connected continuum, is *quasi-monotone* provided that if K is any continuum in B with a non-empty interior relative to B, then $f^{-1}(K)$ has just a finite number of components and each of these maps onto B under f. A result of Wallace [9] together with (10.4) gives at once

(10.41) Corollary. *On a locally connected continuum A quasi-interiority is equivalent to quasi-monotoneity.*

11. INVERSION OF LOCAL CONNECTEDNESS.

(11.1) Theorem. *If A is locally compact, separable and metric, the mapping $f(A) = B$ is interior and K is any continuum in B, then any compact component of $f^{-1}(K)$ maps onto K.*

Proof. Let H be a compact component of $f^{-1}(K)$. Evidently $f(H) \subset K$. Now let V be any neighborhood of H. Choose an open set U so that $\bar{U}$ is compact, $\bar{U}$ is contained in V, and $\mathrm{Fr}(U) \cdot f^{-1}(K) = 0$. Let $f^{-1}(K) \cdot U = W$. Then W is open in $f^{-1}(K)$; and $f(W)$ is open in K, since f is interior. Also W is compact and hence $f(W)$ is compact. Thus $f(W)$ is closed in K. Therefore $f(W) = K$. Whence, $f(H) = K$. For if $f(H)$ were a proper subset of K, so also would be $f(W)$ for V sufficiently near H.

(11.2) Theorem. *Let $f(A) = B$ be light where A and B are locally compact, separable, and metric. Then for any compact subsets K of B and H of A and any*