MATHEMATICS AND STATISTICS

FOR USE IN PHARMACY, BIOLOGY AND CHEMISTRY

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INTRODUCTION

THE Council of the Pharmaceutical Society of Great Britain have authorised the publication of a volume on mathematics and statistics to meet the need of those who practise and study pharmacy for a book which is comprehensive but which will not require for its study an unduly high proportion of their time. The pharmacy student must have reached a high standard in chemistry, physics and biology before commencing to study pharmaceutical subjects, and may not have had the time to acquire a knowledge of mathematics to an equivalent level. In preparing the book, Dr. Saunders and Dr. Fleming have assumed that the reader's knowledge of mathematics is elementary, and have based the text on their experience in training students for the Honours Degree of Bachelor of Pharmacy of the University of London. An approach of this type should make the book acceptable also to those who have already qualified or graduated but who find their knowledge of mathematics inadequate.

By allocating five out of the fifteen chapters to the subject of statistics, the authors acknowledge the growing realisation of the value of statistics in providing a quantitative assessment of the significance of results freed from subjective bias, not only in research but in many routine operations. It is this part of the book that is likely to be most appreciated by many of the older readers.

Although primarily produced for pharmacists and students of pharmacy, the book should be of value to those working and studying in many other branches of pure and applied science, including medicine, biology and chemistry.

February, 1957

PREFACE

MANY students enrolling for courses in pharmacy and other biological subjects have had no training in mathematics beyond the level previously required for the Matriculation Examination. This means, for example, that they know nothing about calculus or about that mysterious quantity e, the base of natural logarithms. Sooner or later in their careers they are certain to encounter such subjects, either in the theory of chemistry or in statistics. In this book we offer a short course in mathematics and statistics which assumes very little knowledge of either topic. We hope that this course will prove helpful to students in schools, technical colleges and universities, and also to practitioners in pharmacy, medicine, biology and chemistry who may wish to brush-up their mathematical knowledge.

Unfortunately, it is almost impossible to write a book on mathematics in the form of an easy narrative, and a certain amount of time and concentration will be required in reading a book of this type. We suggest that readers should go through each chapter twice, the first time to get a general idea of the subject matter, and the second time to follow the arguments in detail, preferably working out the proofs and examples on paper for themselves. Finally, an attempt should be made to solve the problems at the end of the chapter, comparing the answers with those given at the end of the book. The problems in statistics in Chapters 11, 12 and 13 can be worked out most readily with the help of an accurate set of square and square root tables, such as Barlow's Tables (see below), although it is possible to solve them without this aid. One of the problems of Chapter 10 requires log-log graph paper, and Problems 2 and 3 of Chapter 12 require arithmetic probability graph paper.

We have not attempted to give a comprehensive set of references, but suggest that the following books, which contain extensive bibliographies, should be consulted for further reading:

G. J. Kynch, *Mathematics for Chemists*. Butterworths Scientific Publications, London, 1955.

O. L. Davies, *Statistical Methods in Research and Production*. Oliver and Boyd Ltd., Edinburgh, 1949.

J. H. Burn, D. J. Finney and L. G. Goodwin, *Biological Standardisation*. Oxford University Press, London, 1950.

A valuable set of square and square root tables for use in statistical calculation is included in

L. J. Comrie (Ed.), *Barlow's Tables*. Spon Ltd., London, 1947.

PREFACE

When accurate values of statistical quantities are required reference should be made to

R. A. Fisher and F. Yates, *Statistical Tables for Biological, Agricultural and Medical Research*. Oliver and Boyd Ltd., Edinburgh, 1945.

We are indebted to Professor Sir Ronald A. Fisher, Cambridge, Dr. Frank Yates, Rothamsted, and to Oliver and Boyd Limited, Edinburgh, for permission to include in Appendix XIII abridged versions of tables from their book.

We thank our colleagues and friends who have helped us by reading the script and by giving us useful suggestions. We particularly thank Dr. J. W. Fairbairn, Dr. J. R. Hodges and Dr. B. A. Wills for helping with Chapter 14 and for supplying us with numerical data from their own experimental work. Finally, we acknowledge the important contribution of Mr. S. C. Jolly, of the Scientific Publications Department of The Pharmaceutical Society of Great Britain, who has arranged the text in a more presentable form than our original script, and to the help and advice received from the staff of the printers.

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CHAPTER 1

Arithmetic

TRADE is said to be responsible for the development of numbering systems. The earliest numerals were represented by notches on a stick or by scratches on a piece of pottery, but these methods of numeration and the earliest numbering systems, including that used by the Romans, were inadequate in that they did not contain the number zero. The earliest records of arithmetic calculations are said to date from about 3400 B.C., and from the beginning the successful application of mathematics to practical problems has been based on accurate arithmetic. The development of slide rules and calculating machines has reduced much of the labour of arithmetic, but there still remains the necessity for checking results, an operation that should never be omitted.

1. Numbers, factors and primes

The 'alphabet' of the decimal system consists of the ten familiar symbols 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. All numbers, however large or small, can be expressed in terms of these symbols, except irrational numbers (see §5).

Most whole numbers, called *integers*, can be expressed as the product of several numbers, called *factors* of the original number. For instance, the integer 12 can be written as 2×6, or $2\times2\times3$, or 4×3, the sign '$\times$' between the numbers indicating multiplication.

Certain integers cannot be split into factors, and these are called *primes*. The smaller primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41 and 43.

2. Zero and infinity

When zero and a number are added together, the answer is equal to the number: thus $4+0 = 4$. Multiplication of a number by zero gives the answer zero: thus $4\times0 = 0$.

If a finite number is divided by a very small number, the *quotient*, i.e. the result of a division, is a very large number, e.g. $1/0{\cdot}000001 = 1$ million. If the very small number is made smaller still so that its value approaches zero, the quotient becomes an extremely large number and approaches the value 'infinity', which is denoted by the symbol '∞'. Thus

$$1/n \to \infty \quad \text{as} \quad n \to 0.$$

The symbol '$\to$' means 'approaches'; in words, the relation just quoted is 'one divided by n approaches infinity as n approaches zero'.

3. Fractions, decimals and negative numbers

A fraction consists of two integers written one above the other, the upper figure being the *numerator* and the lower the *denominator*. Division of the numerator by the denominator results in a decimal, and if the quotient is less than 1, the symbol zero should always be written in front of the decimal point, e.g. $\frac{1}{4} = 0{\cdot}25$.

The minus sign '$-$' is always written in front of a negative number, but the positive sign '$+$' is often omitted. The multiplication of two positive numbers or two negative numbers always results in a positive product: thus $2\times 2 = 4$ and $(-2)\times(-2) = 4$. The multiplication of a positive number and a negative number always results in a negative product: thus $2\times(-2) = -4$. It follows, therefore, that the square of any number, either positive or negative, is always positive.

If several factors are multiplied together and there is an even number of negative factors, the product will be positive. If there is an odd number of negative factors, the product will be negative: thus

$$(-2)\times 4\times 3\times(-4)\times(-1) = -96,$$

since there are three negative factors.

4. Roots

The square root of a number is the quantity which when multiplied by itself is equal to the original number. For example, the square root of 4 is 2, or it may be -2, since either of these numbers multiplied by itself gives 4. Square roots are denoted by the symbol '$\sqrt{}$'; thus $\sqrt{4}$ equals ± 2, the symbol '$\pm$' meaning plus or minus.

A cube root of a number is the quantity which when multiplied by itself to three factors gives the original number. Cube roots are denoted by the symbol '$\sqrt[3]{}$'; for example $\sqrt[3]{8}$ equals 2, since $2\times 2\times 2 = 8$.

The *n*th root of a number is the quantity which when multiplied by itself to *n* factors gives the original number. It is denoted by the symbol '$\sqrt[n]{}$'. Thus if a equals $\sqrt[n]{x}$, then

$$a\times a\times a \ldots \text{to } n \text{ factors} = x.$$

5. Indices

The limitations of the ten basic symbols of arithmetic become immediately apparent when the number of molecules in 1 gramme-molecule (Avogadro's number) is written down; it is

$$602{,}000{,}000{,}000{,}000{,}000{,}000{,}000.$$

Such numbers are cumbersome and are better expressed by using indices. As has been explained in § 1, numbers other than primes can be expressed as the products of smaller numbers called factors. When these factors are identical, the number of such factors can be indicated by an index number written as a superscript to the right of the factor. For example

$$32 = 2\times2\times2\times2\times2 = 2^5.$$

Expressed in words, 2^5 is 'two raised to the power of five', or simply as 'two to the fifth'. Similarly, Avogadro's number can be written as 602×10^{21}, or more usually as $6{\cdot}02\times10^{23}$. It should be noted that a number raised to the power of one is equal to the number; thus $10^1 = 10$.

A number raised to a negative integral index, i.e. a negative whole number, is equal to the reciprocal of the number raised to an equal positive index: thus

$$2^{-5} = \frac{1}{2^5} = \frac{1}{32}.$$

A fractional index indicates the root of the number, e.g. $8^{1/3}$ means the cube root of 8.

Decimal indices also indicate that a root is taken: thus

$$32^{0\cdot2} = 32^{1/5} = \sqrt[5]{32} = 2.$$
$$2^{0\cdot6} = 2^{3/5} = (2^{1/5})^3 = (\sqrt[5]{2})^3.$$

Indices in multiplication and division. The product of a number raised to a power and the same number raised to another power is the number raised to the sum of the two powers: thus

$$2^3\times2^5 = (2\times2\times2)(2\times2\times2\times2\times2) = 256 = 2^8, \quad \text{i.e.} \quad 2^{(3+5)}.$$

The quotient of a number raised to a power and the same number raised to another power is the number raised to the difference of the two powers: thus

$$\frac{2^5}{2^3} = \frac{2\times2\times2\times2\times2}{2\times2\times2} = 2\times2 = 2^2, \quad \text{i.e.} \quad 2^{(5-3)}.$$

If the denominator has a higher index than has the numerator, the quotient has a negative index: thus

$$\frac{2^3}{2^5} = \frac{2\times2\times2}{2\times2\times2\times2\times2} = \frac{1}{2\times2} = \frac{1}{2^2} = 2^{-2}, \quad \text{i.e.} \quad 2^{(3-5)}.$$

Any number raised to zero power is equal to 1, for

$$\frac{2^3}{2^3} = \frac{8}{8} = 1, \quad \text{or} \quad \frac{2^3}{2^3} = 2^{(3-3)} = 2^0.$$

A positive number raised to any power (positive or negative) gives a positive answer: thus

$$2^4 = 16; \quad 2^{-4} = \frac{1}{2^4} = \frac{1}{16}.$$

A negative number raised to an even integral power (positive or negative) gives a positive answer: thus

$$(-2)^4 = 16; \quad (-2)^{-4} = \frac{1}{(-2)^4} = \frac{1}{16}.$$

A negative number raised to an odd integral power (positive or negative) gives a negative answer: thus

$$(-2)^3 = -8; \quad (-2)^{-3} = \frac{1}{(-2)^3} = -\frac{1}{8}.$$

It is convenient at this point to digress from the properties of indices to define irrational and rational numbers. An *irrational number* is a number whose value cannot be found exactly, e.g. $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, e, π, etc. A *rational number* is a number whose value can be expressed exactly.

Use of indices for expressing large and small numbers. It has already been shown that a very large number like Avogadro's number can be expressed in a compact way by using indices. Similarly, very small numbers can be expressed as a number between 1 and 10 multiplied by 10 raised to a negative integral power. For example, the wavelength of the green line in the mercury spectrum is 0·00005461 cm, and using indices this can be conveniently expressed as $5{\cdot}461 \times 10^{-5}$ cm.

In expressing very large or very small numbers in the form $a \times 10^n$, where n is a positive or negative integer and a is a number between 1 and 10, the value of n can be found by applying the following rules:

(i) For very large numbers, n is a positive integer one less in value than the number of figures before the decimal point: thus

$$602{,}300 = 6{\cdot}023 \times 10^5, \quad \text{i.e.} \quad n = 6-1.$$

(ii) For very small numbers, n is a negative integer one greater in value than the number of successive noughts immediately after the decimal point in the number: thus

$$0{\cdot}0073423 = 7{\cdot}3423 \times 10^{-3}, \quad \text{i.e.} \quad n = -(2+1).$$

Conversely, to write in full numbers expressed in the form $a \times 10^n$, it follows that if n is positive the number of figures before the decimal point will be $n+1$, and if n is negative the number of successive noughts immediately following the decimal point will be $-n-1$.

Calculations involving very large numbers or very small numbers are simplified if the numbers are expressed as a product of a number between 1 and 10 and 10 raised to the appropriate power.

EXAMPLE 1. 1 *gramme-molecule of a gas at* 0°C *and* 760 *mm pressure occupies a volume of* 22·4 *litres. How many molecules of the gas will there be in a* 1-*litre flask evacuated to a pressure of* 10^{-4} *mm at* 0°C?

22·4 litres of gas at 0°C and 760 mm pressure contains

$$6 \times 10^{23} \text{ molecules.}$$

Hence, 1 litre of gas at 0°C and 760 mm pressure contains

$$\frac{6\times10^{23}}{22\cdot4}\text{ molecules,}$$

and 1 litre of gas at 0°C and 10^{-4} mm pressure contains

$$\frac{6\times10^{23}\times10^{-4}}{22\cdot4\times760}\text{ molecules.}$$

The most reliable method for evaluating an expression of this type is to write each quantity as a number between 1 and 10 multiplied by 10 raised to the appropriate power, and then to rearrange the expression so that the numbers between 1 and 10 are in one group and the tens raised to the various powers are in another group: thus

$$\frac{6\times10^{23}\times10^{-4}}{22\cdot4\times760}=\frac{6\times10^{23}\times10^{-4}}{2\cdot24\times10\times7\cdot6\times10^{2}}=\frac{6}{2\cdot24\times7\cdot6}\cdot\frac{10^{23}\times10^{-4}}{10\times10^{2}}.$$

The first factor is worked out [or evaluated using logarithm tables or a slide rule (see §§ 6, 8)]. The second factor is evaluated by adding together the powers of 10. Thus

$$\frac{6}{2\cdot24\times7\cdot6}\cdot\frac{10^{23}\times10^{-4}}{10\times10^{2}}=0\cdot352\times10^{(23-4-1-2)}=0\cdot352\times10^{16}=3\cdot52\times10^{15}.$$

EXAMPLE 2. *A stock solution contains* 5 *g of a drug in* 100 *ml. What volume of this solution should be diluted with water to give* 20 *ml of a solution containing* 10 *microgrammes of the drug in* 1 *ml?*

Since 0·05 g of the drug is contained in 1 ml of stock solution, the volume of this solution containing 1 microgramme (i.e. 1×10^{-6} g) will be

$$1\times10^{-6}/0\cdot05\text{ ml.}$$

To make 20 ml of the required solution, 200 microgrammes of the drug are required. The volume of the stock solution containing 200 microgrammes of drug will be

$$\frac{200\times10^{-6}}{0\cdot05}=\frac{2\times10^{2}\times10^{-6}}{5\times10^{-2}}=\frac{2}{5}\times10^{(2-6+2)}$$

$$=0\cdot4\times10^{-2}=4\times10^{-3}\text{ or }0\cdot004\text{ ml.}$$

EXAMPLE 3. *Calculate the charge, in electrostatic units (e.s.u.), on a single ion of a monovalent metal, given that* (*i*) 96,500 *coulombs are required to deposit* 1 *gramme-equivalent of the metal from solution,* (*ii*) 1 *gramme-equivalent contains* 6×10^{23} *ions, and* (*iii*) 1 *e.s.u. is equivalent to* $3\cdot3\times10^{-10}$ *coulomb.*

The quantity of electricity associated with 1 gramme-equivalent of the metal is 96,500 coulombs. Therefore, the quantity associated with 1 ion is $96{,}500/6\times10^{23}$ coulombs. As $3\cdot3\times10^{-10}$ coulomb is equivalent to 1 e.s.u., 1 coulomb is equivalent to $1/(3\cdot3\times10^{-10})$ e.s.u. Therefore, the charge on a single ion, in e.s.u., is

$$\frac{96{,}500}{6\times10^{23}}\cdot\frac{1}{3\cdot3\times10^{-10}}=\frac{9\cdot65}{6\times3\cdot3}\cdot\frac{10^{4}}{10^{23}\times10^{-10}}$$

$$=0\cdot487\times10^{(4-23+10)}$$

$$=0\cdot49\times10^{-9}=4\cdot9\times10^{-10}.$$

6. Logarithms

The logarithm of a number is the power to which another number, called the base of the logarithm, is raised to become equal to the original number. Thus $10^2 = 100$, and, therefore, according to the above definition, the logarithm of 100 to the base 10 is 2. This is written $\log_{10} 100 = 2$, the base of the logarithm being shown as a subscript; expressed algebraically, if $M = a^x$, then $\log_a M = x$.

Systems of logarithms. Two systems of logarithms are in general use: common logarithms and natural logarithms. They differ in the number used as the base. Common logarithms are based on the number 10 and are used for all arithmetic calculations. They are usually signified by the symbol 'log', no subscript being used. Natural, or Naperian, logarithms are based on a quantity denoted by the letter 'e' and are signified by the symbol 'ln' or '$\log_e$'. The quantity e is an irrational number which can be expressed as a sum of a series. The particular properties of e and natural logarithms are described in Chap. 4.

Common logarithms. Ordinary mathematical tables give the common logarithms of numbers between 1 and 10 to four places of decimals. For more accurate work logarithms to five and seven places of decimals have been published.

The logarithm of a number consists of two parts: an integral part called the *characteristic* and a decimal part called the *mantissa*. The mantissa, which is the quantity given by the logarithm tables, is always positive. The characteristic may be positive or negative, depending upon whether the number is greater or less than unity, and is written down by inspecting the number and applying the following rules:

(i) If the number is greater than 1, the characteristic is a positive integer one less in value than the number of figures before the decimal point. For example, the number 2371 has four figures in front of the decimal point and its logarithm has, therefore, a characteristic of 3. From logarithm tables the mantissa is found to be 0·3749, and so $\log 2371 = 3{\cdot}3749$. This is the power to which the base 10 must be raised to equal 2371, i.e. $10^{3\cdot3749} = 2371$.

(ii) If the number is less than 1, the characteristic is a negative integer, one greater in value than the number of successive noughts immediately after the decimal point. For example, the number 0·00271 has two successive noughts immediately after the decimal point, and $\log 0{\cdot}00271$ has, therefore, a characteristic of -3, written $\bar{3}$ by convention and called 'bar three'. The mantissa of 2·71, obtained from the logarithm tables, is 0·4330, so that $\log 0{\cdot}00271 = \bar{3}{\cdot}4330$.

The device of separating the logarithm into a characteristic and mantissa equivalent to dividing the number into two factors, one of which is 10 raised to an integral power and the other is a number between 1 and 10 whose logarithm can be found directly from logarithm tables. Thus

$$2371 = 10^3 \times 2{\cdot}371 = 10^3 \times 10^{0\cdot3749} = 10^{(3+0\cdot3749)};$$

$$0{\cdot}00271 = 10^{-3} \times 2{\cdot}71 = 10^{-3} \times 10^{0\cdot4330} = 10^{(-3+0\cdot4330)}.$$

To convert a logarithm back into an ordinary number, i.e. to find the antilogarithm of a logarithm, the reverse procedure to that described above is used. The number corresponding to the mantissa is found from the antilogarithm tables and then multiplied by 10 raised to the integral power indicated by the characteristic. For example, antilog 1·1057 is found by looking up 0·1057 in the antilogarithm tables and multiplying the figure 1·276 thus obtained by the figure indicated by the characteristic, i.e. 10^1. Thus antilog $1{\cdot}1057 = 10^1 \times 1{\cdot}276 = 12{\cdot}76$. Similarly, antilog $\bar{2}{\cdot}6136 = 10^{-2} \times 4{\cdot}108 = 0{\cdot}04108$. Since the mantissa is always a positive decimal, i.e. greater than 0 and less than 1, and since $0 = \log 1$ and $1 = \log 10$, the antilogarithm of the mantissa is always a number between 1 and 10.

Logarithms in multiplication and division. From the rules of indices already outlined in § 5, it follows that

(i) the logarithm of the *product* of two numbers is equal to the *sum* of the logarithms of the numbers, and
(ii) the logarithm of the *quotient* of two numbers is equal to the *difference* of the logarithms of the numbers.

Logarithms simplify the processes of multiplication and division by converting these processes into addition and subtraction respectively.

EXAMPLE. *Evaluate by logarithms* 217·2 × 0·0075/33·71.

Inspection of logarithm tables gives the logarithms: thus

$$\log 217{\cdot}2 = 2{\cdot}3369,\ \log 0{\cdot}0075 = \bar{3}{\cdot}8751 \text{ and } \log 33{\cdot}71 = 1{\cdot}5277.$$

Therefore

$$\begin{aligned}\log \frac{217{\cdot}2 \times 0{\cdot}0075}{33{\cdot}71} &= \frac{\log 10^{2\cdot3369} \times 10^{\bar{3}\cdot8751}}{10^{1\cdot5277}} \\ &= \log 10^{(2\cdot3369-3+0\cdot8751-1\cdot5277)} \\ &= \log 10^{\bar{2}\cdot6843} \\ &= \bar{2}{\cdot}6843.\end{aligned}$$

Therefore

$$217{\cdot}2 \times 0{\cdot}0075/33{\cdot}71 = \text{antilog } \bar{2}{\cdot}6843 = 0{\cdot}04834.$$

Logarithms in the evaluation of roots and powers. The use of logarithms to find the powers and roots of a number is based on the rules of indices already outlined in § 5. These rules are illustrated algebraically in Chap. 2, but their practical utilisation in arithmetic is discussed here.

The logarithm of the *n*th power of a number is equal to *n* times the logarithm of the number.

EXAMPLE. *Evaluate* $(2{\cdot}31)^5$.

$$\begin{aligned} \text{Log } 2{\cdot}31 &= 0{\cdot}3636 \\ \log (2{\cdot}31)^5 &= 5\times 0{\cdot}3636 = 1{\cdot}8180 \\ \text{antilog } 0{\cdot}8180 &= 6{\cdot}577 \\ \text{antilog } 1{\cdot}8180 &= 65{\cdot}77. \end{aligned}$$

Therefore

$$(2{\cdot}31)^5 = 65{\cdot}77.$$

Similarly, the logarithm of the nth root of a number is equal to $1/n$ times the logarithm of the number. In calculating the roots of numbers less than 1, it is convenient to obtain the negative characteristic of the answer directly as an integer, and this can be done by rewriting the logarithm so that its negative characteristic is exactly divisible by n. This procedure is illustrated in the following example.

EXAMPLE. *Evaluate* $\sqrt[4]{0{\cdot}00217}$.

$$\begin{aligned} \log \sqrt[4]{0{\cdot}00217} &= \tfrac{1}{4}\log 0{\cdot}00217 = \tfrac{1}{4}\times\bar{3}{\cdot}3365 \\ &= \tfrac{1}{4}(\bar{4}+1{\cdot}3365) = \bar{1}+0{\cdot}3341. \\ \text{antilog } 0{\cdot}3341 &= 2{\cdot}158 \\ \text{antilog } \bar{1}{\cdot}3341 &= 0{\cdot}2158. \end{aligned}$$

Therefore

$$\sqrt[4]{0{\cdot}00217} = 0{\cdot}2158.$$

7. Logarithmic scales, pH and pK

The concentration of hydrogen ions in very pure water is 0·0000001, or 10^{-7}, gramme-equivalent per litre. The hydrogen-ion concentration of a solution is of great importance in, for instance, biochemistry, since quite small changes in this quantity have large effects on the net electrical charge carried by a biocolloid. A change of hydrogen-ion concentration from 10^{-6} to 10^{-4} gramme-equivalent per litre will result in a complete reversal of the electrical charge on an albumen molecule in solution. Sørensen, a Danish scientist, developed a pH scale in order to express these small concentrations in a convenient way. The symbol 'p' simply means −log, and 'H' the hydrogen-ion concentration: thus

$$\text{pH} = -\log\,[\text{H}^+],$$

where $[\text{H}^+]$ is the hydrogen-ion concentration in gramme-equivalents per litre. If $[\text{H}^+] = 0{\cdot}0000001$, i.e. 10^{-7}, then

$$\text{pH} = -\log 10^{-7} = -(-7\log 10) = -(-7\times 1) = 7.$$

The pH scale provides a system whereby the hydrogen-ion concentration of a liquid can be conveniently expressed in terms of small numbers. The usual pH range lies between 0 and 14, since the hydrogen-ion concentration in a normal solution of a strong acid is approximately 1 gramme-equivalent per litre, i.e. pH = 0, and that of a normal solution of sodium hydroxide is approximately 10^{-14} gramme-equivalent per litre, i.e.

pH = 14. The logarithmic nature of the pH scale means that if the pH of a solution is increased by one unit the hydrogen-ion concentration is diminished to one-tenth of its original value.

A similar scale, the pK scale, is used to express the ionic dissociation constants (K values) of weak electrolytes: thus

$$pK = -\log K.$$

pH and pK values are usually non-integral and some care is necessary in calculating them from $[H^+]$ and K values.

As an example, the dissociation constant K of acetic acid is $1{\cdot}75 \times 10^{-5}$ at 20°. Therefore

$$pK = -\log(1{\cdot}75 \times 10^{-5}) = -(\log 1{\cdot}75 + \log 10^{-5}) = -(\log 1{\cdot}75 - 5)$$
$$= 5 - \log 1{\cdot}75 = 5 - 0{\cdot}2430 = 4{\cdot}76.$$

The pK value is less in magnitude than the negative index of 10 in the constant K.

Non-integral values of pH are calculated in a similar way. Suppose the hydrogen-ion concentration of a solution is 2×10^{-4} gramme-equivalent per litre. Then

$$pH = -\log(2 \times 10^{-4}) = -(\log 2 - 4) = 4 - \log 2 = 4 - 0{\cdot}3010 = 3{\cdot}70.$$

8. Slide rules

The processes of multiplication, division, extraction of square roots, obtaining the powers of numbers, etc., are very easily carried out mechanically by using the slide rule. This instrument consists of several scales, the

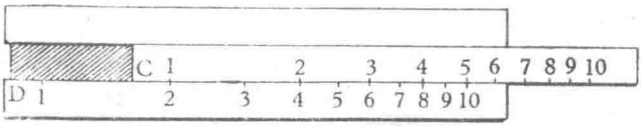

FIG. 1a. Multiplication by the slide rule

most important of which are the two adjacent identical scales, each graduated with numbers from 1 to 10. The numbers occur in ascending order from left to right and are spaced logarithmically, for example, the space between 2 and 4 is the same as that between 4 and 8, since

$$\log 4 - \log 2 = \log 8 - \log 4.$$

Fig. 1a shows these two scales. The lower scale D is fixed and the upper scale C is engraved on a central sliding section so that it moves parallel to D.

To multiply 2 by 4, the number 1 on the sliding scale C is set opposite to the 2 on the lower fixed scale, and the answer to this multiplication is the