Mathematical Surveys and Monographs

Volume 225

Expanding Thurston Maps

Mario Bonk

Daniel Meyer

American Mathematical Society

This monograph is devoted to the study of the dynamics of expanding Thurston maps under iteration. A Thurston map is a branched covering map on a two-dimensional topological sphere such that each critical point of the map has a finite orbit under iteration. It is called expanding if, roughly speaking, preimages of a fine open cover of the underlying sphere under iterates of the map become finer and finer as the order of the iterate increases.

Every expanding Thurston map gives rise to a fractal space, called its visual sphere. Many dynamical properties of the map are encoded in the geometry of this visual sphere. For example, an expanding Thurston map is topologically conjugate to a rational map if and only if its visual sphere is quasisymmetrically equivalent to the Riemann sphere. This relation between dynamics and fractal geometry is the main focus for the investigations in this work.

ISBN 978-0-8218-7554-4

9 780821 875544

SURV/225

SURV 225

Expanding Thurston Maps

Bonk and Meyer

AMS

Mathematical
Surveys
and
Monographs

Volume 225

Expanding Thurston Maps

Mario Bonk
Daniel Meyer

American Mathematical Society
Providence, Rhode Island

2010 *Mathematics Subject Classification*. Primary 37-02, 37F10, 37F20, 30D05, 30L10.

For additional information and updates on this book, visit
www.ams.org/bookpages/surv-225

Library of Congress Cataloging-in-Publication Data

Names: Bonk, Mario. | Meyer, Daniel, 1969–
Title: Expanding Thurston maps / Mario Bonk, Daniel Meyer.
Description: Providence, Rhode Island: American Mathematical Society, [2017] | Series: Mathematical surveys and monographs; volume 225 | Includes bibliographical references and index.
Identifiers: LCCN 2017017476 | ISBN 9780821875544 (alk. paper)
Subjects: LCSH: Algebraic topology. | Mappings (Mathematics) | AMS: Dynamical systems and ergodic theory – Research exposition (monographs, survey articles). msc | Dynamical systems and ergodic theory – Complex dynamical systems – Polynomials; rational maps; entire and meromorphic functions. msc | Dynamical systems and ergodic theory – Complex dynamical systems – Combinatorics and topology. msc | Functions of a complex variable – Entire and meromorphic functions, and related topics – Functional equations in the complex domain, iteration and composition of analytic functions. msc | Functions of a complex variable – Analysis on metric spaces – Quasiconformal mappings in metric spaces. msc
Classification: LCC QA612 .B66 2017 | DDC 515/.39–dc23
LC record available at https://lccn.loc.gov/2017017476

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at `http://www.ams.org/`

10 9 8 7 6 5 4 3 2 1 22 21 20 19 18 17

Expanding Thurston Maps

Mario Bonk
Daniel Meyer

Preface

This book is the result of an intended research paper that grew out of control. A preprint containing a substantial part of our investigations was already published on arXiv in 2010. To make its content more accessible, we decided to include some additional material. These additions more than doubled the size of this work as compared with the 2010 version and caused a long delay in its completion.

More than fifteen years ago we became both interested in some basic problems on quasisymmetric parametrization of 2-spheres. This is related to the dynamics of rational maps—an observation we believe was first made by Rick Kenyon. During our time at the University of Michigan we decided to join forces and to investigate this connection systematically.

We realized that for the relevant rational maps an explicit analytic expression is not so important, but rather a geometric-combinatorial description. As this became our preferred way of looking at these objects, it was a natural step to consider a more general class of maps that are not necessarily holomorphic. The relevant properties can be condensed into the notion of an *expanding Thurston map*, which is the topic of this book. We will discuss the underlying ideas more thoroughly in the introduction (Chapter 1).

Part of this work overlaps with studies by other researchers, notably Haïssinsky-Pilgrim [**HP09**], and Cannon-Floyd-Parry [**CFP07**]. We would like to clarify some of the interrelations of our investigations with these works. Theorem 15.1 (in the body of the text) was announced by the first author during an Invited Address at the AMS Meeting at Athens, Ohio, in March 2004, where he gave a short outline of the proof. After the talk he was informed by Bill Floyd and Walter Parry that related results had been independently obtained by Cannon-Floyd-Parry (which later appeared as [**CFP07**]).

Theorem 18.1 (ii) was previously published by Haïssinsky-Pilgrim as part of a more general statement [**HP09**, Theorem 4.2.11]. Special cases go back to work by the second author [**Me02**] and unpublished joint work by Bruce Kleiner and the first author. The current, more general version emerged after a visit of the first author at the University of Indiana at Bloomington in February 2003.

During this visit the first author explained to Kevin Pilgrim concepts of quasiconformal geometry and his joint work with Bruce Kleiner on Cannon's conjecture in geometric group theory. Kevin Pilgrim in turn pointed out Theorem 11.1 and the ideas for its proof to the first author. After this visit versions of Theorem 18.1 (ii) with an outline for the proof were found independently by Kevin Pilgrim and the first author. A proof of Theorem 18.1 (ii) was discovered soon afterwards by the authors using ideas from [**Me02**] (see [**Me10**] for an argument along similar lines) in combination with Theorem 15.1.

We are indebted to many people. Conversations with Bruce Kleiner, Peter Haïssinsky, and Kevin Pilgrim have been especially fruitful. We would also like to thank Jim Cannon, Bill Floyd, Lukas Geyer, Misha Hlushchanka, Zhiqiang Li, Dimitrios Ntalampekos, Walter Parry, Juan Souto, Dennis Sullivan, and Mike Zieve for various useful comments. Two anonymous referees provided us with valuable feedback. Their considerable efforts were very much appreciated.

Qian Yin was so kind to let us incorporate parts of her thesis. We are grateful to Jana Kleineberg for her careful proofreading and her help with some of the pictures. We are also happy to acknowledge the patient support of our editors from the American Mathematical Society, Ed Dunne and Ina Mette.

Over the years we received funding from various sources. Mario Bonk was partially supported by NSF grants DMS 0244421, DMS 0456940, DMS 0652915, DMS 1058283, DMS 1058772, DMS 1162471, and DMS 1506099. Daniel Meyer was partially supported by an NSF postdoctoral fellowship, the Deutsche Forschungsgemeinschaft (DFG-ME 4188/1-1), the Academy of Finland, projects SA-134757 and SA-118634, and the Centre of Excellence in Analysis and Dynamics Research, project No. 271983.

Los Angeles and Liverpool, March 2017

Notation

We summarize some of the most important notation used in this book for easy reference.

When an object A is defined to be another object B, we write $A := B$ for emphasis.

We denote by $\mathbb{N} = \{1, 2, \dots\}$ the set of natural numbers and by $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ the set of natural numbers including 0. We write $\mathbb{Z}$ for the set of integers, and $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ for the set of rational, real, and complex numbers, respectively. For $k \in \mathbb{N}$, we let $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$ be the cyclic group of order k.

We also consider $\widehat{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. Given $a, b \in \widehat{\mathbb{N}}$ we write $a|b$ if a divides b. This notation is extended to $\widehat{\mathbb{N}}$-valued functions. If $A \subset \widehat{\mathbb{N}}$, then $\mathrm{lcm}(A) \in \widehat{\mathbb{N}}$ denotes the least common multiple of the numbers in A. See Section 2.5 for more details.

The *floor* of a real number x, denoted by $\lfloor x \rfloor$, is the largest integer $m \in \mathbb{Z}$ with $m \leq x$. The *ceiling* of a real number x, denoted by $\lceil x \rceil$, is the smallest integer $m \in \mathbb{Z}$ with $x \leq m$.

The symbol $\boldsymbol{i}$ stands for the imaginary unit in the complex plane $\mathbb{C}$. The real and imaginary part of a complex number z are indicated by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively, and its complex conjugate by $\overline{z}$. The open unit disk in $\mathbb{C}$ is denoted by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and the open upper half-plane by $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

We let $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. It carries the *chordal metric* σ given by formula (A.5) (in the appendix). Similarly, we let $\widehat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. Here we consider $\widehat{\mathbb{R}}$ as a subset of $\widehat{\mathbb{C}}$, and so $\widehat{\mathbb{R}} \subset \widehat{\mathbb{C}}$.

The *Lebesgue measure* on $\mathbb{R}^2$, $\mathbb{C}$, $\widehat{\mathbb{C}}$, or $\mathbb{D}$ is denoted by $\mathcal{L}$. If necessary, we add a subscript here to avoid ambiguities. More precisely, $\mathcal{L} = \mathcal{L}_{\mathbb{R}^2}$ and $\mathcal{L} = \mathcal{L}_{\mathbb{C}}$ are the Euclidean area measures on $\mathbb{R}^2$ and $\mathbb{C}$, $\mathcal{L} = \mathcal{L}_{\widehat{\mathbb{C}}}$ is the spherical area measure on $\widehat{\mathbb{C}}$, and $\mathcal{L} = \mathcal{L}_{\mathbb{D}}$ the hyperbolic area measure on $\mathbb{D}$ considered as the hyperbolic plane.

When we consider two objects A and B, and there is a natural identification between them that is clear from the context, we write $A \cong B$. For example, $\mathbb{R}^2 \cong \mathbb{C}$ if we identify a point $(x, y) \in \mathbb{R}^2$ with $x + y\boldsymbol{i} \in \mathbb{C}$.

The derivative of a holomorphic function f is denoted by f' as usual. If $\Omega \subset \widehat{\mathbb{C}}$ is an open set and $f \colon \Omega \to \widehat{\mathbb{C}}$ is a holomorphic map, then $f^\sharp$ stands for its *spherical derivative* (see (A.6)). For a differentiable (not necessarily holomorphic) map, we use Df to denote its derivative considered as a linear map between suitable tangent spaces. If these tangent spaces are equipped with norms, then we let $\|Df\|$ be the operator norm of Df. Sometimes we use subscripts here to indicate the norms.

Two non-negative quantities a and b are said to be *comparable* if there is a constant $C \geq 1$ (possibly depending on some ambient parameters) such that

$$\frac{1}{C}a \leq b \leq Ca.$$

We then write $a \asymp b$. The constant C is referred to as $C(\asymp)$. Similarly, we write $a \lesssim b$ or $b \gtrsim a$, if there is a constant $C > 0$ such that $a \le Cb$, and refer to the constant C as $C(\lesssim)$ or $C(\gtrsim)$. If we want to emphasize the parameters $\alpha, \beta, \dots$ on which C depends, then we write $C(\asymp) = C(\alpha, \beta, \dots)$, etc.

The cardinality of a set X is denoted by $\#X$ and the identity map on X by id_X. If $x_n \in X$ for $n \in \mathbb{N}$ are points in X, we denote the sequence of these points by $\{x_n\}_{n\in\mathbb{N}}$, or just by $\{x_n\}$ if the index set $\mathbb{N}$ is understood.

If $f\colon X \to X$ is a map and $n \in \mathbb{N}$, then

$$f^n := \underbrace{f \circ \cdots \circ f}_{n \text{ factors}}$$

is the n-th iterate of f. We set $f^0 := \mathrm{id}_X$ for convenience, but unless otherwise indicated it is understood that $n \in \mathbb{N}$ if we speak of an iterate f^n of f.

Let $f\colon X \to Y$ be a map between sets X and Y. If $U \subset X$, then $f|U$ stands for the *restriction* of f to U. If $A \subset Y$, then $f^{-1}(A) := \{x \in X : f(x) \in A\}$ is the preimage of A in X. Similarly, $f^{-1}(y) := \{x \in X : f(x) = y\}$ is the preimage of a point $y \in Y$.

If $f\colon X \to X$ is a map, then preimages of a set $A \subset X$ or a point $p \in X$ under the n-th iterate f^n are denoted by $f^{-n}(A) := \{x \in X : f^n(x) \in A\}$ and $f^{-n}(p) := \{x \in X : f^n(x) = p\}$, respectively.

Let (X, d) be a metric space, $a \in X$, and $r > 0$. By $B_d(a, r) = \{x \in X : d(a, x) < r\}$ we denote the open and by $\overline{B}_d(a, r) = \{x \in X : d(a, x) \le r\}$ the closed ball of radius r centered at a. If $A, B \subset X$, we let $\mathrm{diam}_d(A)$ be the diameter, $\overline{A}$ be the closure of A in X, and

$$\mathrm{dist}_d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$$

be the distance of A and B. If $p \in X$, we let $\mathrm{dist}_d(p, A) := \mathrm{dist}_d(\{p\}, A)$. For $\epsilon > 0$,

$$\mathcal{N}_{d,\epsilon}(A) := \{x \in X : \mathrm{dist}_d(x, A) < \epsilon\}$$

is the *open ϵ-neighborhood* of A with respect to d. If $\gamma\colon [0, 1] \to X$ is a path, we denote by $\mathrm{length}_d(\gamma)$ the length of γ. Given $Q \ge 0$, we denote by $\mathcal{H}^Q_d$ the *Q-dimensional Hausdorff measure* on X with respect to d. We drop the subscript d in our notation for $B_d(a, r)$, etc., if the metric d is clear from the context. For the Euclidean metric on $\mathbb{C}$ we sometimes use the subscript $\mathbb{C}$ for emphasis. So, for example,

$$B_{\mathbb{C}}(a, r) := \{z \in \mathbb{C} : |z - a| < r\}$$

denotes the Euclidean ball of radius $r > 0$ centered at $a \in \mathbb{C}$.

The *Gromov product* of two points $x, y \in X$ with respect to a basepoint $p \in X$ in a metric space X is denoted by $(x \cdot y)_p$ or by $(x \cdot y)$ if the basepoint p is understood (see Section 4.2). The *boundary at infinity* of a Gromov hyperbolic space X is represented by $\partial_\infty X$. If a group G acts on a space X, then we write $G \curvearrowright X$ to indicate this action.

Often we use the notation $I = [0, 1]$. If X and Y are topological spaces, then a *homotopy* is a continuous map $H\colon X \times I \to Y$. For $t \in I$, we let $H_t(\cdot) := H(\cdot, t)$ be the *time-t map* of the homotopy.

The symbol S^2 indicates a 2-sphere, which we think of as a topological object. Similarly, T^2 is a topological 2-torus. For a 2-torus with a Riemann surface structure we write $\mathbb{T}$ (see Section A.8).

Often S^2 (or the Riemann sphere $\widehat{\mathbb{C}}$) is equipped with certain metrics that induce its topology. The *visual metric* induced by an expanding Thurston map f is usually denoted by ϱ (see Chapter 8). The *canonical orbifold metric* of a rational Thurston map f is indicated by ω_f (see Section A.10).

The (topological) *degree* of a branched covering map f between surfaces is denoted by $\deg(f)$ and the *local degree* of f at a point x by $\deg_f(x)$ or $\deg(f, x)$ (see Section 2.1). We write $\operatorname{crit}(f)$ for the *set of critical points* of a branched covering map (see Section 2.1), and $\operatorname{post}(f)$ for the set of *postcritical points* of a Thurston map f (see Section 2.2).

The *ramification function* of a Thurston map f is denoted by α_f (see Definition 2.7), and the *orbifold* associated with f by $\mathcal{O}_f$ (see Definition 2.10).

For a given Thurston map $f\colon S^2 \to S^2$ we usually use the symbol $\mathcal{C}$ to indicate a Jordan curve $\mathcal{C} \subset S^2$ that satisfies $\operatorname{post}(f) \subset \mathcal{C}$.

When we consider objects that are defined in terms of the n-th iterate of a given Thurston map, then we often use the upper index "n" to emphasize this.

For a topological cell c in a topological space $\mathcal{X}$ we denote by ∂c the *boundary* of c, and by $\operatorname{int}(c)$ the *interior* of c (see Section 5.1). Note that ∂c and $\operatorname{int}(c)$ usually do not agree with the boundary or interior of c as a subset of $\mathcal{X}$.

Cell decompositions of a space $\mathcal{X}$ are usually denoted by $\mathcal{D}$ (see Chapter 5). Let $n \in \mathbb{N}_0$, $f\colon S^2 \to S^2$ be a Thurston map, and $\mathcal{C} \subset S^2$ be a Jordan curve with $\operatorname{post}(f) \subset \mathcal{C}$. We then write $\mathcal{D}^n(f, \mathcal{C})$ for the cell decomposition of S^2 consisting of the *cells of level* n or n*-cells* defined in terms of f and $\mathcal{C}$ (see Definition 5.14). The set of corresponding n*-tiles* is denoted by $\mathbf{X}^n$, the set of n*-edges* by $\mathbf{E}^n$, and the set of n*-vertices* by $\mathbf{V}^n$ (see Section 5.3).

In this context we often "color" tiles "black" or "white". We then use the subscripts $\mathtt{b}$ and $\mathtt{w}$ to indicate the color (see the end of Section 5.3). For example, the black and white 0-tiles are denoted by $X^0_{\mathtt{b}}$ and $X^0_{\mathtt{w}}$, respectively.

The n*-flower* of an n-vertex v is denoted by $W^n(v)$ (see Section 5.6). The number $D_n = D_n(f, \mathcal{C})$ is the minimal number of n-tiles required to join opposite sides (see (5.15)).

The number $m(x, y) = m_{f,\mathcal{C}}(x, y)$ is defined in Definition 8.1. The *expansion factor* of a visual metric is usually denoted by Λ (see Definition 8.2).

We write $\Lambda_0(f)$ for the *combinatorial expansion factor* of a Thurston map f (see Proposition 16.1).

The *topological entropy* of a map f is denoted by $h_{top}(f)$, and the *measure-theoretic entropy* of f with respect to a measure μ by $h_\mu(f)$. The *measure of maximal entropy* of an expanding Thurston map f is indicated by ν_f. See Chapter 17 for these concepts.

For a rational Thurston map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ we write Ω_f for its *canonical orbifold measure* (see Section A.10) and, if f is also expanding, λ_f for the unique probability measure on $\widehat{\mathbb{C}}$ that is absolutely continuous with respect to Lebesgue measure (see Chapter 19).

Contents