INTERMEDIATE CLASSICAL MECHANICS

Joseph Norwood, Jr.

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PREFACE

My reason for writing a book on classical mechanics at the intermediate (undergraduate) level when so many good texts are already on the market has to do with the conviction that classical mechanics should be taken to include not only particle mechanics and a bit on rigid bodies, but also many-body systems often neglected in texts at this level. Consequently, the present text treats elastic bodies, and also many-body systems from a fluid and statistical point of view. There is a considerable amount of material on hydromagnetic and plasma systems including a discussion of waves in hydromagnetic and hydrodynamic media from the elegant vantage point afforded by the theory of characteristics introduced early in the text. Other features worth noting include a comprehensive discussion of vectors, a treatment of adiabatic invariance, a thorough discussion of rocket motion, Lagrangian and Hamiltonian formulations via the calculus of variations, a chapter on stability, and a modern treatment of relativity including the general theory.

Appropriate problems have been included at the end of each section rather than lumped at the end of the chapter. The book as a whole is suitable for either a one or two-semester course with the first six chapters comprising the first semester. The level of mathematics is such that the two-semester course of elementary calculus is prerequisite and the course of ordinary differential equations is corequisite.

The author would like to thank his typist who also doubles as his wife and the mother of four tolerant children, for her hard work and support during the preparation of this book. Special thanks go to Professor Carl Adler of East Carolina University for a critical reading of the manuscript. Appreciation is also due to the reviewers and my editor Logan Campbell for many helpful suggestions and comments.

JOSEPH NORWOOD, JR.

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Chapter
One

PRINCIPLES OF NEWTONIAN MECHANICS

1-1 The Nature of Mechanics

Mechanics, the study of the reaction of massive bodies to forces imposed on them, was the first exact science developed by man. The early Greeks dealt with questions concerning mechanics, but the answers obtained were based, for the most part, on esthetic arguments rather than experimental evidence. Clearly, the success of any theory generated in this way can only be qualitative at best, and even this marginal triumph requires luck. The chance of an exact agreement to within the limits of experimental error is vanishingly small. It was not until the time of Aristotle that the necessity for interplay between theory and experiment began to become apparent. In his *De Generatione et Corruptione* Aristotle notes:[1]

> Lack of experience diminishes our power of taking a comprehensive view of the admitted facts. Hence, those who dwell in intimate association with nature and its phenomena grow more and more able to formulate, as the foundation of their theories, principles such as to admit of a wide and coherent development; while those whom devotion to abstract discussions has rendered unobservant of the facts are too ready to dogmatize on the basis of a few observations.

Unfortunately, Aristotle did not always practice what he preached in this regard, and it was not until the time of Galileo that the *scientific method* finally became established.

[1] Aristotle, *De Generatione et Corruptione*, book I, cap. II, H. H. Joachim (tr.). Oxford: Clarendon Press, 1922.

The history of mechanics is closely associated with man's interest in, and ideas about, the motion of the heavenly bodies. Greek science had developed a geocentric cosmology wherein the sun, stars, and planets all revolved in circular orbits with the earth stationary and at the center of the universe. This cosmology became dogmatized by the Church and was not seriously challenged until Copernicus published his heliocentric theory 17 centuries after the time of Aristotle. Galileo, whose career began about 40 years after the publication of Copernicus' theory, was an outspoken critic of Aristotle's doctrines. The first to utilize the telescope for astronomical observations, Galileo began to amass evidence against the geocentric cosmology of Aristotle. The adverse reaction of the Church, however, caused him to become cautious in his later work and to publish in the form of dialogues. These dialogues presented both sides of the argument in such a way that the fallacies of the conventional view would be obvious to an openminded reader without the necessity for Galileo to commit himself directly in print. He wrote two of these dialogues: *Dialogues on the Ptolemaic and Copernican Systems* published in 1632 and *Dialogues on Two New Sciences* (motion and cohesion) published in 1636. The dialogues on motion report Galileo's experiments on accelerating bodies and contain his mature deliberations on their significance. He states, for example, that all bodies fall at the same speed if the resistance of the medium can be neglected, and he establishes that the path of a projectile is parabolic under certain conditions.[2] His work on mechanics exerted a profound influence on Newton, born in 1642, the same year that Galileo died.

Although apparently a slow starter as a student, Newton's genius asserted itself shortly after he entered Cambridge in 1661; he discovered the binomial theorem, developed properties of infinite series, and was one of the inventors of differential calculus while still a student. The plague that swept Europe in the 1660s caused the university to close its doors at various periods. According to Newton, it was during this time, which was spent at home on his mother's estate at Woolsthorpe, that the idea of universal gravitation was developed.[3] Newton had read the work of Kepler, a contemporary of Galileo, while at Cambridge. Kepler was a theoretician who had taken the precise planetary observations of his mentor, Tycho Brahe, and had deduced three empirical laws of planetary motion from these data. He recognized in a qualitative way the presence of a universal binding force for planetary systems but was unable to take the critical step to formulate a theory. Newton sought to find a law of attraction between two massive bodies—for instance, the sun and a planet —such that Kepler's third law, which states that the square of the period of rotation is proportional to the cube of the mean distance separating the two

[2] Galileo understood that forces serve as mechanical agents, but he did not manage to establish a quantitative connection between force and motion.

[3] The "falling apple" incident supposedly occurred during this period.

bodies, is obtained as a result. Newton found that a gravitational attraction that varies as the inverse square of the separation distance would produce this relation. He attempted to test this inverse-square law by calculating the measurable ratio of the acceleration of the moon toward the earth to the acceleration of falling bodies at the earth's surface. It was not until Newton managed to prove that homogeneous spheres attract one another as though all their mass were concentrated at the centers that this test was successful. Through his law of universal gravitation, Newton could now derive all three of Kepler's empirical laws.

In 1687 the Royal Society published Newton's *Principia*, in which he expounded his ideas on mechanics. The first two of the three parts of this work are concerned with establishing the foundations of Newton's mechanics; the third part presents a detailed treatment of planetary motion. The three laws of motion on which classical mechanics are based appear for the first time in this book. In formulating these laws, Newton had to break considerable new ground; he was the first, for instance, to distinguish between weight and mass.

A broad range of phenomena can be described directly on the basis of Newton's laws. This textbook is concerned primarily with such phenomena. Certain types of problems are more amenable to treatment by the use of alternative formulations of Newton's theory that are due to Lagrange and Hamilton. If the speeds attained by the bodies of interest are allowed to approach the speed of light or if very large masses or distances are contemplated, then the theory of relativity developed by Einstein in the years 1904 through 1916 must be used. These departures from pure Newtonian mechanics are presented in the closing chapters as introductory material to a more advanced treatment that is beyond the scope of this book.

Kinematics is the branch of mechanics that deals with the classification and description of the types of motions experienced by massive bodies. In many of the problems with which we will deal, the net force acting on the body vanishes. Such a condition is referred to as a state of *equilibrium* and can be further classified as *static* if the velocity of the body is zero or *stationary* if the velocity is constant but nonzero. The question of the *stability* of equilibria will also be considered. In other cases, we shall be interested in bodies that move under the action of forces; this branch of mechanics is called *dynamics*. The bodies whose mechanical state we will examine include point masses (particles), systems of particles, systems of many particles (gases), fluids, elastic bodies, and rigid bodies.

1-2 Scalars, Vectors, and Tensors

The concept of a *field* is almost universal in physics. A field can be regarded as a mathematical idealization of some physical phenomenon involving the notion of extension. The simplest fields can be specified by assigning a single *measure number* at each point of space, as, for example, the temperature distribution in an extended body. Fields of this type are known as *scalars*. Other fields descriptive of common phenomena require three measure numbers at each point of space and are called *vector* fields.[4] Scalars and vectors are special cases of an elegant class of functions known as *tensors*. Scalars are tensors of zero *rank* and vectors are tensors of the first rank.

The special property of tensors that makes them so uniquely useful in formulating physics is their invariance under coordinate transformation. Clearly, it is desirable for a physical theory to be formulated in such a way as to be independent in its results of the choice of coordinate system. This choice should be made purely on the basis of convenience. For instance, a temperature distribution may be written in one coordinate system as $T(x, y, z)$ and in terms of another coordinate system as $T(x', y', z')$. The property that distinguishes a scalar like T from a nonscalar function of the coordinates is that for scalars

$$T(x, y, z) = T(x', y', z'). \tag{1-1}$$

A vector is similarly distinguished from a nonvector function by its invariance under coordinate transformation. In order to ascertain the mathematical criteria for this invariance, let us examine the changes in the coordinates of a point as the coordinate system is rotated about an axis passing through its origin. In Fig. 1-1 we show a point P from the point of view of

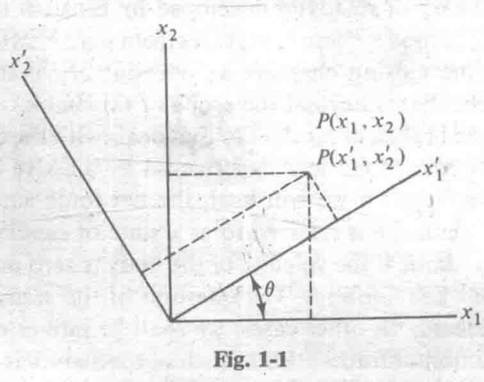

Fig. 1-1

two coordinate systems rotated through an angle θ with respect to one another. The axes are labeled x_1, x_2 instead of x, y in order to facilitate the use of summation notation. By simple trigonometry, we see that the primed

[4] In spaces having a dimensionality higher than three, more measure numbers at each point are required, the number being equal to the dimensionality of the space in question. In the present discussion and throughout most of the book we shall only be concerned with the three-dimensional Euclidean space.

coordinates of P are given in terms of the unprimed coordinates and the angle θ by

$$\begin{aligned} x_1' &= x_1 \cos\theta + x_2 \sin\theta, \\ x_2' &= -x_1 \sin\theta + x_2 \cos\theta. \end{aligned} \tag{1-2}$$

Let us introduce

$$\lambda_{ij} = \cos(x_i', x_j) \tag{1-3}$$

for the cosine of the angle between the x_i' and x_j axes. From Fig. 1-1 we see that

$$\begin{aligned} \lambda_{11} &= \cos(x_1', x_1) = \cos\theta, \\ \lambda_{12} &= \cos(x_1', x_2) = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \\ \lambda_{21} &= \cos(x_2', x_1) = \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta, \\ \lambda_{22} &= \cos(x_2', x_2) = \cos\theta, \end{aligned} \tag{1-4}$$

in terms of which Eq. (1-2) can be written

$$\begin{aligned} x_1' &= \lambda_{11}x_1 + \lambda_{12}x_2, \\ x_2' &= \lambda_{21}x_1 + \lambda_{22}x_2. \end{aligned} \tag{1-5}$$

The extension to three dimensions is perfectly straightforward:

$$\begin{aligned} x_1' &= \lambda_{11}x_1 + \lambda_{12}x_2 + \lambda_{13}x_3, \\ x_2' &= \lambda_{21}x_1 + \lambda_{22}x_2 + \lambda_{23}x_3, \\ x_3' &= \lambda_{31}x_1 + \lambda_{32}x_2 + \lambda_{33}x_3. \end{aligned} \tag{1-6}$$

Equation (1-6) can be written much more compactly in summation notation as

$$x_i' = \sum_{j=1}^{3} \lambda_{ij}x_j, \qquad i = 1, 2, 3. \tag{1-7}$$

The symmetry of this equation allows the inverse transformation to be written by inspection:

$$x_i = \sum_{j=1}^{3} \lambda_{ji}x_j', \qquad i = 1, 2, 3. \tag{1-8}$$

The λ_{ij} are often written as a square array called a *matrix*, where λ denotes the matrix

$$\lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix}. \tag{1-9}$$

Let us examine the properties of this matrix. In Fig. 1-2 we have a line of length R drawn in an arbitrary direction from the origin of a rectangular coordinate system to the point P. The direction of OP is described by the angles α, β, γ between OP and the x_1, x_2, x_3 axes, respectively. If the coordinate axes are mutually orthogonal, then $R_{x_1}^2 + R_{x_2}^2 + R_{x_3}^2 = R^2$ by virtue of the theorem of Pythagorus. Since $R_{x_1} = R\cos\alpha$ and similarly for the x_2 and x_3 components, the *direction cosines* must be related according to

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \tag{1-10}$$

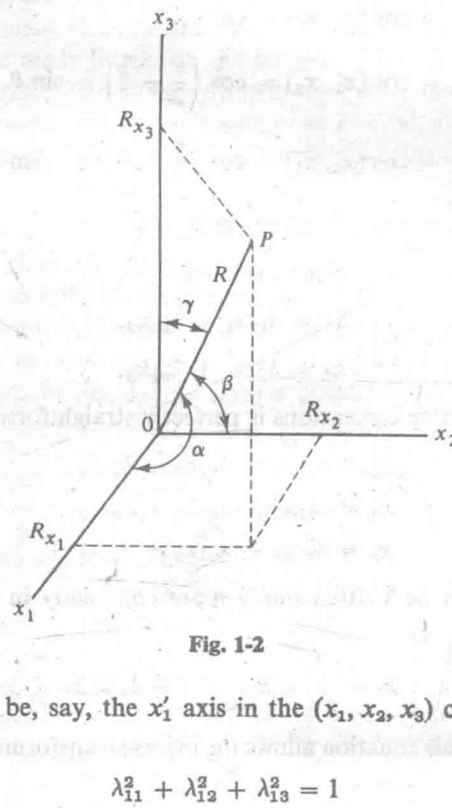

Fig. 1-2

If we take OP to be, say, the x_1' axis in the (x_1, x_2, x_3) coordinate system, we see that

$$\lambda_{11}^2 + \lambda_{12}^2 + \lambda_{13}^2 = 1$$

and similarly for the x_2' and x_3' axes. This result can be expressed in summation notation as

$$\sum_j \lambda_{ij}\lambda_{kj} = 1, \qquad i = k, \tag{1-11}$$

which amounts to three relations between the nine elements of the λ matrix.

Next, consider two lines drawn from the origin of a rectangular coordinate system such that the lines are characterized by direction cosines $(\cos\alpha_1,$

$\cos\beta_1, \cos\gamma_1)$ and $(\cos\alpha_2, \cos\beta_2, \cos\gamma_2)$, respectively. The cosine of the angle θ between these lines is then given by

$$\cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2. \tag{1-12}$$

In terms of, say, the x_1' and x_2' axes in the (x_1, x_2, x_3) coordinate system, Eq. (1-12) can be expressed in terms of the matrix elements as

$$\cos\frac{\pi}{2} = 0 = \lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} + \lambda_{13}\lambda_{23}$$

and we are led to a second general relation

$$\sum_j \lambda_{ij}\lambda_{kj} = 0, \qquad i \neq k. \tag{1-13}$$

Equations (1-11) and (1-13) can be combined into a single summation equation

$$\sum_j \lambda_{ij}\lambda_{kj} = \delta_{ik}, \tag{1-14}$$

where δ_{ik} is the *Kronecker delta* symbol, defined by

$$\delta_{ik} = \begin{cases} 0, & i \neq k, \\ 1, & i = k. \end{cases} \tag{1-15}$$

The six relations represented by Eq. (1-14), which is called the *orthogonality condition*, are based on the fact that the coordinate axes in each of the coordinate systems are mutually perpendicular.

The λ matrix given in Eq. (1-9) is a *square matrix*—that is, it has an equal number of rows and columns. Such is not true of all matrices, however. For instance, we can write the three measure numbers that define the coordinates of a point as either a *column matrix*

$$\mathsf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{1-16}$$

or a *row matrix*

$$\mathsf{x} = (x_1 \quad x_2 \quad x_3). \tag{1-17}$$

Next, we need to establish the rules describing the multiplication of two matrices. These rules have, in effect, already been established by the requirement that they be consistent with Eq. (1-7) when x_i and x_i' are written in matrix form. Thus we find that

$$\mathsf{x}' = \lambda\mathsf{x} = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{1-18}$$

must be equivalent to

$$\begin{aligned} x_1' &= \lambda_{11}x_1 + \lambda_{12}x_2 + \lambda_{13}x_3, \\ x_2' &= \lambda_{21}x_1 + \lambda_{22}x_2 + \lambda_{23}x_3, \\ x_3' &= \lambda_{31}x_1 + \lambda_{32}x_2 + \lambda_{33}x_3. \end{aligned} \tag{1-19}$$

This establishes the multiplication rule for the special case of the product of a 3 × 3 matrix and a 3 × 1 matrix. For the more general case where the number of rows of the first matrix and the number of columns of the second matrix are arbitrary, the product rule is

$$R_{ij} = \sum_k P_{ik}Q_{kj} \tag{1-20}$$

for the product $\mathsf{R} = \mathsf{PQ}$. We see that in order for the product QP to be defined, Q must have the same number of columns as P has rows. A particularly interesting aspect of matrix multiplication is that the product is, in general, noncommutative. That is,

$$\mathsf{PQ} \neq \mathsf{QP}. \tag{1-21}$$

Indeed, unless the two matrices are both square, QP is not even defined.

The *transpose* of a matrix P is accomplished by interchanging rows and columns—that is,

$$\tilde{P}_{ij} = P_{ji}, \tag{1-22}$$

where $\tilde{\mathsf{P}}$ denotes the transpose of the matrix P. If we consider two matrices, P of order $(n \times h)$ and Q of order $(h \times m)$, then the product PQ is defined and $\mathsf{R} = \mathsf{PQ}$ is of order $(n \times m)$. Consider the transposed matrices. The transpose of P is of order $(h \times n)$ and $\tilde{\mathsf{Q}}$ is of order $(m \times h)$. Thus $\tilde{\mathsf{P}}\tilde{\mathsf{Q}}$ is undefined and we find for $\tilde{\mathsf{R}}$

$$\tilde{\mathsf{R}} = \tilde{\mathsf{Q}}\tilde{\mathsf{P}}. \tag{1-23}$$

In general, if $\mathsf{F} = \mathsf{ABCD}\ldots\mathsf{X}$, then $\tilde{\mathsf{F}} = \tilde{\mathsf{X}}\ldots\tilde{\mathsf{D}}\tilde{\mathsf{C}}\tilde{\mathsf{B}}\tilde{\mathsf{A}}$.

The *unit matrix* or *identity matrix* is given by

$$\mathsf{E} = \delta_{ij}, \tag{1-24}$$

where δ_{ij} is the Kronecker delta. So it is a matrix, all of whose off-diagonal elements are zero, with the diagonal elements all equal to one. This matrix leaves any matrix with which it is multiplied in either order unchanged:

$$\mathsf{EP} = \mathsf{PE} = \mathsf{P}. \tag{1-25}$$

The λ matrix with which we have been concerned in this section enjoys the property that

$$\lambda = \tilde{\lambda}^{-1} \tag{1-26}$$

or

$$\lambda\tilde{\lambda} = \tilde{\lambda}\lambda = \mathsf{E} \tag{1-27}$$