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V.I. Arnold

Geometrical Methods in the Theory of Ordinary Differential Equations

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Preface

Newton's fundamental discovery, the one which he considered necessary to keep secret and published only in the form of an anagram, consists of the following: *Data aequatione quotcunque fluentes quantitae involvente fluxiones invenire et vice versa.* In contemporary mathematical language, this means: "It is useful to solve differential equations".

At present, the theory of differential equations represents a vast conglomerate of a great many ideas and methods of different nature, very useful for many applications and constantly stimulating theoretical investigations in all areas of mathematics.

Many of the routes connecting abstract mathematical theories to applications in the natural sciences lead through differential equations. Many topics of the theory of differential equations grew so much that they became disciplines in themselves; problems from the theory of differential equations had great significance in the origins of such disciplines as linear algebra, the theory of Lie groups, functional analysis, quantum mechanics, etc. Consequently, differential equations lie at the basis of scientific mathematical philosophy (Weltanschauung).

In the selection of material for this book, the author intended to expound basic ideas and methods applicable to the study of differential equations. Special efforts were made to keep the basic ideas (which are, as a rule, simple and intuitive) free from technical details. The most fundamental and simple questions are considered in the greatest detail, whereas the exposition of the more special and difficult parts of the theory has been given the character of a survey.

The book begins with the study of some special differential equations integrable by quadrature. The main Attention is paid mainly to connections with general mathematical ideas, methods, and concepts (resolution of singularities, Lie groups, and Newton diagrams), on the one hand, and to applications to the natural sciences on the other, rather than to the formal cookbook aspect of the elementary theory of integration.

The theory of partial differential equations of the first order is considered by means of the natural contact structure in the manifold of 1-jets of functions. The necessary elements of the geometry of contact structures are developed incidentally, making the entire theory independent of other sources.

A significant portion of the book is concerned with methods which are usually called *qualitative*. The recent development of the qualitative theory of differential equations, originated by Poincaré, led to the realization that similar to the fact that the explicit integration of differential equations is generally impossible, the qualitative study of general differential equations with a multi-dimensional phase space turns out to be impossible. The book discusses the analysis of differential equations from the point of view of structural stability, that is, the stability of the qualitative picture with respect to a small change in the differential equations. The basic results obtained after the first publications of Andronov and Pontrjagin in this area are expounded: the elements of the theory of structurally stable Anosov systems, all trajectories of which are exponentially unstable, and Smale's theorem on the nondensity of structurally stable systems. We also discuss the significance of these mathematical discoveries to applications. (We speak of the description of stable chaotic regimes of motion like turbulence.)

The most powerful and frequently applicable methods of study of differential equations are the various asymptotic methods. We develop the basic ideas of the averaging method going back to the work of the founders of celestial mechanics and widely usable in all those areas of application, where a slow evolution has to be separated from fast oscillations (Bogoljubov, Mitropol'skiĭ, and others).

In spite of the abundant research in averaging, in the problem of evolution even for the simplest multi-frequency systems, everything is not entirely clear. We give a survey of the work concerning passage through resonances and capture to resonance in an attempt to illuminate the problem.

The basis of the averaging method is the idea of annihilating perturbations by means of an appropriate choice of the coordinate system. This very idea lies at the basis of the theory of Poincaré normal forms. The method of normal forms is the fundamental method of the local theory of differential equations, which describes the behavior of phase curves in the neighborhood of a singular point or a closed phase curve. In this book, we describe the main results of the method of Poincaré normal forms, including a proof of Siegel's fundamental theorem on the linearization of a holomorphic mapping.

Important applications of the method of Poincaré normal forms come across not only in the study of a single differential equation, but also in bifurcation theory, where the subject of research is a family of equations depending on parameters.

Bifurcation theory studies the qualitative change under the variation of the parameters on which the system depends. For general values of the parameters, we usually have to deal with generic systems (all singular points are simple, etc.). However, if a system depends on parameters, then for some values of the parameters we cannot avoid degeneracies (for example, the fusion of two singular points of a vector field).

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In a one-parameter system, we generically encounter only simple degeneracies (those which we cannot get rid of by a small perturbation of the family). Consequently, there arises a hierarchy of degeneracies according to the codimensions of the corresponding surfaces in the function space of all systems under study: in one-parameter generic families, only degeneracies corresponding to surfaces of codimension 1 occur, and so on.

Recent progress in bifurcation theory is connected with the application of ideas and methods of the general theory of singularities of differentiable mappings due to Whitney.

This book concludes with a chapter on bifurcation theory, in which the methods developed in the preceding chapters are applied, and main results obtained in this field, beginning with the fundamental work of Poincaré and Andronov, are described.

In discussing all of these subjects, the author attempts to avoid the axiomatic-deductive style, with its unmotivated definitions concealing the fundamental ideas and methods; similar to parables, they are explained only to disciples in private.

The axiomization and algebraization of mathematics, after more than 50 years, has led to the illegibility of such a large number of mathematical texts that the threat of complete loss of contact with physics and the natural sciences has been realized. The author attempts to write in such a way that this book can be read by not only mathematicians, but also all users of the theory of differential equations.

We only assume a little general mathematical knowledge on the part of the reader, let us say, roughly the first two courses of a university program; for example, familiarity with the textbook V. I. Arnold, Ordinary Differential Equations, Moscow, Nauka, 1974 [in English, Cambridge, MA, MIT Press, 1973, 1978]* is sufficient (but not necessary).

The exposition is developed in such a way that the reader can omit passage that turn out to be difficult for him, without much harm to the understanding of what follows: as much as possible, we avoid references from one chapter to another, and even from one paragraph to another.

The content of this book constitutes the material of a series of mandatory and special courses delivered by the author at the Department of Mechanics and Mathematics of Moscow State University, 1970–1976 to students of mathematics in grade II–III, and to mathematicians working in applications.

The author expresses his gratitude to students O. E. Hadin, A. K. Kobal'dži, E. M. Kaganovaja, and to Professor Ju. S. Il'jašenko, whose notes were very useful in the preparation of this book. The notes of a special course composed by Il'jašenko and the notes of the lectures given in the experimental group have been in the department library for a number of

* In the exposition of some special questions, we have also used or recalled elementary information on differential forms, Lie groups, and functions of a complex variable. This information is not necessary for the understanding of most of the book.

years. The author is grateful to the many readers and students of these courses for a series of valuable remarks used in the preparation of the book. The author is grateful to referees D. V. Anosov and V. A. Pliss for a careful and helpful review of the manuscript.

June, 1977 V. Arnold

Notation

$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{Z}$	the set of integers
$\mathbb{R}^n$	the n-dimensional real linear space
$\exists$	there exists
$\forall$	for every
$a \in A$	the element a of the set A
$A \subset B$	the subset A of the set B
$A \cap B$	intersection of the sets A and B
$A \cup B$	union of the sets A and B
$A \backslash B$	difference of the sets A and B (the part of A outside B)
$A \times B$	direct product of the sets A and B (the set of pairs (a, b), $a \in A$, $b \in B$)
$A \oplus B$	direct sum of linear spaces
$f: A \to B$	a mapping f of A into B
$x \mapsto y$ or $y = f(x)$	the mapping f maps the element x onto the element y
$\operatorname{Im} f$ or $f(A)$	image under the mapping f (but $\operatorname{Im} z$ is the imaginary part of z)
$f^{-1}(y)$	complete inverse image of the point y under the mapping f (the set of all x for which $f(x) = y$)
$\operatorname{Ker} f$	kernel of the linear operator f (the complete inverse image of zero)
$\dot{f}$	rate of change of the function f (derivative with respect to time t)
$f', f_*, df/dx, Df/Dx$	derivative of the mapping f
$T_x M$	the tangent space of the manifold M at the point x
$A \Rightarrow B$	assertion A implies B
$A \Leftrightarrow B$	assertions A and B are equivalent
$\omega_1 \wedge \omega_2$	exterior product of the differential forms ω_1 and ω_2
$f \circ g$	composition of mappings $[(f \circ g)(x) = f(g(x))]$
$L_v f$	derivative of the function f in the direction of the vector field v

Contents

Chapter 1

Special Equations

In the study of differential equations, methods from all fields of mathematics are used. In this chapter, we discuss selected special equations and types of equation. Special attention is paid, on the one hand, to the significance in applications of these equations and, on the other hand, to the connection between research methods and various general mathematical problems (resolution of singularities, Newton diagrams, Lie groups of symmetries, etc.). This chapter concludes with the elementary theory of the one-dimensional stationary Schrödinger equation and the geometric theory of a nonlinear equation of the second order.

§ 1. Differential Equations Invariant under Groups of Symmetries

In this section, general arguments are discussed on which the methods of integration of differential equations in explicit form are based. As an example, we discuss the theory of similarity, i.e., the theory of homogeneous and quasihomogeneous equations.

A. Groups of Symmetries of Differential Equations

Let us consider a vector field v in a phase space U.

Definition. A diffeomorphism $g\colon U \to U$ is called a *symmetry* of v if it transforms v into itself:

$$v(gx) = g_{*x}v(x).$$

Then v is said to be *invariant* under g.

Examples

1. A vector field with components independent of x on the (x, y)-plane is invariant under translations along the x-axis (Fig. 1).

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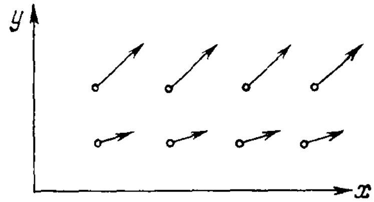

Figure 1.

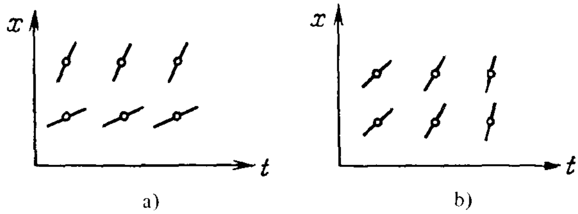

Figure 2.

2. The vector field $x\partial_x + y\partial_y$ on the Euclidean (x, y)-plane is invariant under the dilations $g(x, y) = (\lambda x, \lambda y)$ and rotations.

All the symmetries of a given field form a group.

Exercise. Determine the symmetry group of the field $x\partial_x + y\partial_y$ in the coordinate plane (x, y).

Let us consider a direction field in the extended phase space.

Definition. A diffeomorphism of the extended phase space is called a *symmetry of the direction field* if it maps the field into itself. The direction field is then said to be *invariant* with respect to this diffeomorphism.

EXAMPLES

1. The direction field of the equation $\dot{x} = v(x)$ is invariant under translations along the t-axis (Fig. 2a).
2. The direction field of the equation $\dot{x} = v(t)$ is invariant under translations along the x-axis (Fig. 2b).

Definition. The differential equation $\dot{x} = v(x)$ (respectively, $\dot{x} = v(x, t)$) is said to be *invariant* under the diffeomorphism g of the phase space (respectively, of the extended phase space) if the vector field v (respectively, the direction field v) is invariant under this diffeomorphism g. The diffeomorphism g is then called a *symmetry* of the given equation.

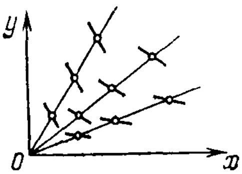

Figure 3.

Theorem. *Any symmetry of a given equation transforms phase (integral) curves of the equation into phase (integral) curves of the same equation.*

◀ Let $x = \varphi(t)$ be a solution of the equation $\dot{x} = v(x)$ and let g be a symmetry. Then $x = g(\varphi(t))$ is also a solution. Consequently, symmetries transform phase curves into phase curves. The proof is analogous for integral curves. ▶

EXAMPLE

The family of integral curves of the equation $\dot{x} = v(t)$ is transformed into itself by translations along the x-axis, and that of the equation $\dot{x} = v(x)$ by translations along the t-axis.

The following examples are frequently encountered in applications under the names "theory of similarity", "comparison of dimensions", or "scaling".

B. Homogeneous Equations

Definition. The direction field in the plane without the origin 0 is said to be *homogeneous* if it is invariant under all the dilations

$$g^\lambda(x, y) = (e^\lambda x, e^\lambda y), \qquad \lambda \in \mathbb{R}.$$

The differential equation $dy/dx = v(x, y)$ is said to be *homogeneous* if its direction field is homogeneous (Fig. 3).

In other words, the directions of the field have to be parallel to each other at all points of the same ray starting from the origin of the coordinate system:

$$v(e^\lambda x, e^\lambda y) \equiv v(x, y).$$

EXAMPLE

The function f is said to be homogeneous of degree d if $f(e^\lambda x, e^\lambda y) \equiv e^{\lambda d} f(x, y)$. Any form (homogeneous polynomial) of degree d can serve as an example. Let P and Q be two forms of degree d depending on the variables x and y. The differential equation

$$\dot{x} = P, \qquad \dot{y} = Q$$

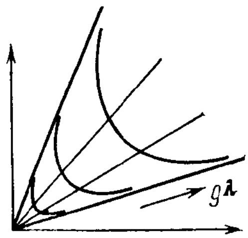

Figure 4.

is given by the vector field (P, Q) in the plane. The corresponding direction field in the domain $P \neq 0$ is the direction field of the homogeneous equation

$$\frac{dy}{dx} = \frac{Q}{P} \quad \left(\text{e.g., } \frac{dy}{dx} = \frac{ax + by}{cx + dy}, \frac{dy}{dx} = \frac{x^2 - y^2}{x^2 + y^2}, \text{etc.}\right)$$

Remark. The domain (of definition) of a homogeneous field does not necessarily have to be the whole punctured plane. Homogeneous fields may be defined in any homogeneous (i.e., invariant under dilations) domains, e.g., in an angular sector with vertex 0, etc.

Theorem. *Every integral curve of an arbitrary homogeneous equation is transformed by any dilation g^λ into an integral curve of the same equation.*

Consequently, given a homogeneous equation, it is sufficient to study only one integral curve in each sector of the plane.

The proof may be obtained by an immediate application of the theorem in § 1A.

Exercise. Let P and Q be forms of degree d. Prove that the phase curves of the system $\dot{x} = P$, $\dot{y} = Q$ are obtained from each other by dilations (Fig. 4).

If one of these curves is closed and has period T, then the dilation g^λ transforms it into another closed phase curve having period $T/e^{\lambda(d-1)}$.

C. Quasi-Homogeneous Equations and the "Comparison of Dimensions"

Let us fix the real numbers α and β and consider the family of transformations

$$g^s(x, y) = (e^{\alpha s}x, e^{\beta s}y) \tag{1}$$

which dilate in the x and y directions by different amounts.

Note that Eq. (1) defines a one-parameter group of linear transformations in the plane (Fig. 5).

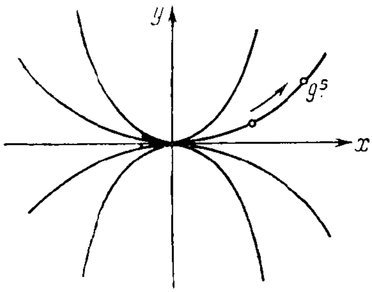

Figure 5.

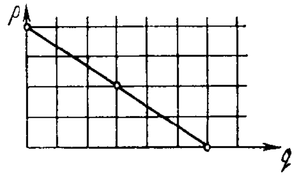

Figure 6.

Definition. The function f is said to be *quasi-homogeneous of degree d* if

$$f(g^s(x, y)) \equiv e^{ds} f(x, y).$$

EXAMPLE

If $\alpha = \beta = 1$, then we obtain the ordinary homogeneous functions of degree d.

Quasi-homogeneous degrees add under the multiplication of functions. They are also called *weights*. For example, x has weight α, y has weight β, x^2y has weight $2\alpha + \beta$, and so on. All quasi-homogeneous monomials of a fixed degree can be easily seen in the following *Newton diagram* (Fig. 6). We identify the monomial $x^p y^q$ with the point (p, q) of the integer lattice. Exponents of all possible monomials of degree d are the integer lattice points on the segment given by the equation $d = \alpha p + \beta q$ in the (p, q)-plane.

Exercise. Choose weights in such a way that the function $x^2 + xy^3$ is quasi-homogeneous.

Definition. The differential equation $dy/dx = v(x, y)$ is said to be *quasi-homogeneous* (*with weights* α and β) if its direction field v is invariant under the transformations in Eq. (1).

From the general theorem of Section 1A on symmetries, we can derive the following.

Theorem. *The integral curves of a quasi-homogeneous equation are obtained from each other by using the transformations of Eq. (1).*

Exercise. Prove that the function $v(x, y)$ is the right-hand side of a quasi-homogeneous differential equation (with weights (α, β)) if and only if it is quasi-homogeneous of degree $d = \beta - \alpha$.

Remark. The above definitions and theorems can be easily extended to the case of more than two variables and to differential equations of order greater than 1. In particular, it is easy to prove the following.

Theorem. *Let* $\gamma: y = y(x)$ *be a curve in the* (x, y)*- plane and let* $d^k y/dx^k = F$ *at the point* (x_0, y_0). *Then for the curve* $g^s\gamma$ *we have*

$$\frac{d^k y}{dx^k} = e^{(\beta - k\alpha)s}F$$

at the corresponding point.

In other words, $d^k y/dx^k$ *is transformed, as is* y/x^k, *by the transformations of Eq.* (1), which in turn explains the convenience of the notation $d^k y/dx^k$.

Exercise. Prove that if a particle in a homogeneous force field of degree d moves along the trajectory Γ in time T, then the same particle moves along the dilated trajectory $\lambda\Gamma$ in time

$$T' = \lambda^{(1-d)/2}T.$$

Solution. The Newtonian equation $d^2x/dt^2 = F(x)$, where F is homogeneous of degree d, is transformed into itself by appropriate transformations of the form of Eq. (1). Namely, it is sufficient to choose weights α (for x) and β (for t) such that $\alpha - 2\beta = \alpha d$. Then $\beta = ((1 - d)/2)\alpha$. Consequently, $T' = \lambda^{(1-d)/2}T$ corresponds to the dilation $x' = \lambda x$.

Exercise. Prove Kepler's third law: The squares of the times needed for similar trajectories in the gravitational field are proportional to the cubes of the linear measurements of these trajectories.

Solution. From the solution of the preceding exercise with $d = -2$ (the law of universal gravity), we obtain $T' = \lambda^{3/2}T$.

Exercise. Determine how the period of oscillation depends on the amplitude in the case of a restoring force proportional to the elongation (linear oscillator) and to the cube of the elongation (weak force).

Answer. In the case of a harmonic oscillator, the period does not depend on the amplitude: in the case of a weak oscillator, it is inversely proportional to the amplitude.

Exercise. It is known that a top with a vertical axis has a critical angular velocity: if the angular velocity is greater than the critical velocity, then the top stands up firmly vertically, and if it is less, it falls.

How does the critical angular velocity change if we take the top to the moon, where the gravitational force is six times less than on the earth?

Answer. It decreases by a factor of $\sqrt{6}$.

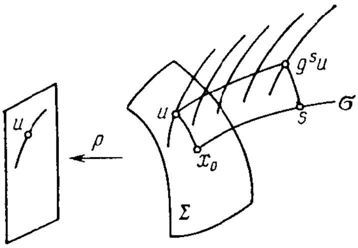

Figure 7.

D. Applications of One-Parameter Groups of Symmetries to Lowering the Order

Theorem. *If a one-parameter group of symmetries of a direction field in $\mathbb{R}^n$ is known, then the problem of integration of the corresponding differential equation reduces to the problem of integrating a differential equation in $\mathbb{R}^{n-1}$.*

In particular, if a one-parameter group of symmetries of a direction field on the plane is known, then the corresponding equation $dy/dx = f(x, y)$ can be integrated explicitly.

◀ Let $\{g^s\}$ be the group of symmetries under consideration. Let us consider the orbits $\{g^s x\}$ of the flow $\{g^s\}$. We can define (at least locally) an $(n - 1)$-dimensional *orbit space* (the quotient space with respect to the action of g^s) and a mapping p of the initial space onto the quotient space (p maps the orbits of the flow $\{g^s\}$ into points). It turns out that the initial direction field is mapped by p into a new direction field on the $(n - 1)$-dimensional orbit space; one only has to integrate it. ▶

More precisely, consider a point $x_0 \in \mathbb{R}^n$ and assume that the orbit of $\{g^s\}$ going through x_0 is a curve σ. Through x_0 let us draw an $(n - 1)$-dimensional local transversal Σ to σ. In the neighborhood of x_0, introduce the local coordinate system (s, u), where the point $g^s u$ of the initial space corresponds to the pair $s \in \mathbb{R}$, $u \in \Sigma$. Then in the neighborhood of x_0, the projection p onto the orbit space and the action of the group g^s of symmetries are given by the formulas

$$p(s, u) = u, \qquad g^{s_1}(s_2, u) = (s_1 + s_2, u)$$

(the points on the surface Σ parameterize the local orbits.)

We note that if the group g^s is given explicitly, then the coordinates (s, u) can be found explicitly. We write the initial differential equation in these coordinates. If our direction field is not tangent to Σ at x_0 (which can always be achieved by the choice of Σ), then in the neighborhood of this point our equation takes the form

$$\frac{du}{ds} = v(s, u).$$

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