时代教育·国外高校优秀教材精选

（英文版）

傅里叶分析

Classical and Modern Fourier Analysis

（美）Loukas Grafakos 著

机械工业出版社
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出 版 说 明

随着我国加入 WTO，国际间的竞争越来越激烈，而国际间的竞争实际上也就是人才的竞争、教育的竞争。为了加快培养具有国际竞争力的高水平技术人才，加快我国教育改革的步伐，国家教育部近来出台了一系列倡导高校开展双语教学、引进原版教材的政策。以此为契机，机械工业出版社陆续推出了一系列国外影印版教材，其内容涉及高等学校公共基础课，以及机、电、信息领域的专业基础课和专业课。

引进国外优秀原版教材，在有条件的学校推动开展英语授课或双语教学，自然也引进了先进的教学思想和教学方法，这对提高我国自编教材的水平，加强学生的英语实际应用能力，使我国的高等教育尽快与国际接轨，必将起到积极的推动作用。

为了做好教材的引进工作，机械工业出版社特别成立了由著名专家组成的国外高校优秀教材审定委员会。这些专家对实施双语教学做了深入细致的调查研究，对引进原版教材提出了许多建设性意见，并慎重地对每一本将要引进的原版教材一审再审，精选再精选，确认教材本身的质量水平，以及权威性和先进性，以期所引进的原版教材能适应我国学生的外语水平和学习特点。在引进工作中，审定委员会还结合我国高校教学课程体系的设置和要求，对原版教材的教学思想和方法的先进性、科学性严格把关。同时尽量考虑原版教材的系统性和经济性。

这套教材出版后，我们将根据各高校的双语教学计划，举办原版教材的教师培训，及时地将其推荐给各高校选用。希望高校师生在使用教材后及时反馈意见和建议，使我们更好地为教学改革服务。

机械工业出版社

序

傅里叶变换是在数字信号处理方面很有用的一个方法，在通信和信息专业有很强的应用。本书总结、整理了近 50 年来调和分析（傅里叶分析）理论研究的基本成果，系统性强，内容先进全面。

作者 Loukas Grafakos，希腊雅典人，在加利福尼亚大学洛杉矶分校获得博士学位，现任密苏里州大学数学教授。曾在耶鲁大学和华盛顿大学（圣路易斯市）任教，也曾在数学科学研究院（伯克利市）和匹兹堡大学做访问学者，曾因出色的教学被授予 Kemper Fellow 奖，自著或与人合著了 40 篇傅里叶方面的文章。

本书内容包括 L^p 空间和插值，极大函数，傅里叶变换以及广义函数，一维环群上的傅里叶分析，卷积型奇异积分，Littlewood-Paley 理论与乘子，光滑性和函数空间，BMO 和 Carleson 测度，非卷积型奇异积分，加权不等式，傅里叶积分的有界性和收敛性。讲述方式易于接受，只要有本科知识就能够阅读，各章节有预备知识提要，习题例题丰富，称得上是一本优秀的教材。最后提供了 574 篇文献目录，读研究人员也很有必要。

北京师范大学数学学院

王昆扬

PREFACE

Prologue

The word *analysis* comes from the Greek $\alpha\nu\acute{\alpha}\lambda\upsilon\sigma\iota s$, which means "dissolving into pieces." This is usually the first step of a process that leads to a careful study and understanding of an object or phenomenon. The antithetical process, called $\sigma\acute{\upsilon}\nu\theta\varepsilon\sigma\iota s$, is equally significant as it assembles the analyzed pieces after they have been individually examined. This procedure is the heart of Fourier analysis. Through its aorta, this heart disseminates information to a variety of applications. Fourier analysis is therefore a prism that diffracts ideas into a rainbow of uses and applications, making the subject one of the richest and most far-reaching in mathematics.

The primary goal of this text is to present the theoretical foundation of the field of Fourier analysis. This book is mainly addressed to graduate students in mathematics and is designed to serve for a three-course sequence on the subject. The only prerequisite for understanding the text is satisfactory completion of a course in measure theory, Lebesgue integration, and complex variables. This book is intended to present the selected topics in some depth and stimulate further study. Although the emphasis falls on real variable methods in Euclidean spaces, a chapter is devoted to the fundamentals of analysis on the torus. This material is included for historical reasons, as the genesis of Fourier analysis can be found in trigonometric expansions of periodic functions in several variables.

The choice of the material in the text reflects a measure of personal taste; however, a certain effort has been made to include a variety of topics of general interest. Much attention is given to details, which are designed to facilitate the understanding of first-time readers. Based on my personal experience, I felt a need to include details related to topics that articles often omit, leaving beginners to struggle through without explanation. Although it will behoove many readers to skim through the more technical aspects of the presentation and concentrate on the flow of ideas, the mere fact that details are here for reference will be comforting to some. I hope that students will profit from this comprehensive presentation and learn how to do mathematics rigorously. Unfortunately, including so many details has led to the large size of the book. But as one's maturity and familiarity with the subject increases, topics slowly become natural and reading is significantly accelerated.

The exercises that follow each section enrich the material of the corresponding section and provide an opportunity to develop additional intuition and deeper comprehension. Some of them are rather rudimentary and require minimal skill, while others are more interesting and challenging. Only a few exercises are considered difficult, but these are given with hints. A special effort has been made to prepare the exercises, which unfortunately did not double, but almost tripled, the amount of time and effort it took to complete this text. I hope that the reader will find this extra effort beneficial.

The historical notes given at the end of each chapter are intended to provide an accurate account of past research but also to suggest directions for further investigation. This book was partly written with the purpose of attracting students to research. Many of the topics in Chapter 10 lead to open problems that have bewildered mathematicians for decades. It is hoped that many students will be fascinated by the easy statements, yet the delicate complexity of some of these problems, and pursue a deeper understanding.

The text is completely self-contained as the appendix includes the miscellaneous material needed throughout. Certain user-friendly conventions have been adopted to facilitate searching. For instance, theorems, propositions, definitions, lemmas, remarks, and examples are numbered according to the order in which they appear in each section. Exercises are numbered similarly and can be easily located.

As this book is intended for a three-course sequence on the subject, I would like to suggest a slowly paced initial breakdown of the material, flexible enough to accommodate adjustments: Semester I: Chapters 1, 2, 3, and 4. Semester II: Chapters 5, 6, 7, and 9. Semester III: Chapters 8, 10, and other topics. Sections or subsections marked by a star would normally be omitted in a yearly course.

I am solely responsible for any misprints, mistakes, and historical omissions in this book. Please contact me directly (loukas@math.missouri.edu) if you have any comments, suggestions, improvements, or corrections. Instructors are also welcome to contact me to obtain further hints on the existing exercises in the text. Suggestions for other exercises are also welcome. A list of current errata with acknowledgements will be kept at the following URL:

http://math.missouri.edu/~loukas/Fourier-Analysis

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I am indebted to Nigel Kalton, who assisted me with the exposition of Sections 1.4, 4.5, and 4.6 and provided me with the proof of Lemma 1.4.20. I have learned a lot of mathematics and I have immensely profited from my numerous conversations with him. I have been very fortunate to interact with him and I thank him sincerely for everything he has taught me.

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Finally, none of this work would have been possible without the constant love and support of my wife Suzanne. I am taking this opportunity to express my deep gratitude to her. The recent arrival of our beautiful children Joanna and Constandina has changed the meaning of my life. It is to Suzanne, Joanna, and the newborn Constandina that this book is dedicated.

Loukas Grafakos
Columbia, Missouri
loukas@math.missouri.edu

April 2003

CONTENTS

CHAPTER 1

L^p Spaces and Interpolation

The primary focus of this monograph is the study of Fourier series and integrals of functions. Many of their properties are quantitatively expressed in terms of the integrability of the function. For this reason it is desirable to acquire a good understanding of spaces of functions whose modulus to a power p is integrable. These are called Lebesgue spaces and are denoted by L^p. Although an in-depth study of Lebesgue spaces falls outside the scope of this book, it seems appropriate to devote a chapter to reviewing some of their fundamental properties.

The emphasis of our review will be basic interpolation between Lebesgue spaces. Many problems in Fourier analysis concern boundedness of operators on Lebesgue spaces and interpolation provides a framework that often simplifies their study. For instance, in order to show that a linear operator maps L^p into itself for all $1 < p < \infty$, it is sufficient to show that it maps the (smaller) Lorentz space $L^{p,1}$ into the (larger) Lorentz space $L^{p,\infty}$ for the same range of p's. Moreover, some further reductions can be made in terms of the Lorentz space $L^{p,1}$. This and other considerations indicate that interpolation is a powerful tool in the study of boundedness of operators.

Although we will be mainly concerned with L^p subspaces of the Euclidean space $\mathbf{R}^n$, we discuss in this chapter L^p spaces of arbitrary measure spaces, as they often present a useful general setting. Moreover, many proofs in the text go through when Lebesgue measure is replaced by a more general measure.

1.1. L^p and Weak L^p

Let X be a measure space and let μ be a positive, not necessarily finite, measure on X. For $0 < p < \infty$, $L^p(X, \mu)$ will denote the set of all complex-valued μ-measurable functions on X whose modulus to the pth power is integrable. $L^\infty(X, \mu)$ will be the set of all complex-valued μ-measurable functions f on X such that for some $B > 0$, the set $\{x : |f(x)| > B\}$ has μ-measure zero. Two functions in $L^p(X, \mu)$ will be considered equal if they are equal μ-almost everywhere. The notation $L^p(\mathbf{R}^n)$ will be reserved for the space $L^p(\mathbf{R}^n, |\cdot|)$, where $|\cdot|$ denotes n-dimensional Lebesgue measure. Lebesgue measure on $\mathbf{R}^n$ will also be denoted by dx. Within context and in the lack of ambiguity, $L^p(X,\mu)$ will simply be L^p. The space $L^p(\mathbf{Z})$ equipped with counting measure will be denoted by $\ell^p(\mathbf{Z})$ or simply ℓ^p.

For $0 < p < \infty$, we define the L^p quasi-norm of a function f by

$$\|f\|_{L^p(X,\mu)} = \left(\int_X |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} \tag{1.1.1}$$

and for $p = \infty$ by

$$\|f\|_{L^\infty(X,\mu)} = \inf \big\{ B > 0 \, : \, \mu(\{x : |f(x)| > B\}) = 0 \big\}. \tag{1.1.2}$$

It is well known that Minkowski's (or the triangle) inequality

$$\|f+g\|_{L^p(X,\mu)} \le \|f\|_{L^p(X,\mu)} + \|g\|_{L^p(X,\mu)} \tag{1.1.3}$$

holds for all f, g in $L^p = L^p(X,\mu)$, whenever $1 \le p \le \infty$. Since in addition $\|f\|_{L^p(X,\mu)} = 0$ implies that $f = 0$ (μ-a.e.), the L^p spaces are normed linear spaces for $1 \le p \le \infty$. For $0 < p < 1$, inequality (1.1.3) is reversed when $f, g \ge 0$. However, the following substitute of (1.1.3) holds:

$$\|f+g\|_{L^p(X,\mu)} \le 2^{(1-p)/p} \big(\|f\|_{L^p(X,\mu)} + \|g\|_{L^p(X,\mu)} \big) \tag{1.1.4}$$

and thus the spaces $L^p(X,\mu)$ are quasi-normed linear spaces. See also Exercise 1.1.5. For all $0 < p \le \infty$, it can be shown that every Cauchy sequence in $L^p(X,\mu)$ is convergent, and hence the spaces $L^p(X,\mu)$ are complete. For the case $0 < p < 1$ we refer to Exercise 1.1.8. Therefore, the L^p spaces are Banach spaces for $1 \le p \le \infty$ and quasi-Banach spaces for $0 < p < 1$. For any $p \in (0,\infty) \setminus \{1\}$ we will use the notation $p' = \frac{p}{p-1}$. Moreover we set $1' = \infty$ and $\infty' = 1$ so that $p'' = p$ for all $p \in (0,\infty]$. Hölder's inequality says that for all $p \in [1,\infty]$ and all measurable functions f, g on (X,μ) we have

$$\|fg\|_{L^1} \le \|f\|_{L^p} \|g\|_{L^{p'}} \, .$$

It is a well-known fact that the dual $(L^p)^*$ of L^p is isometric to $L^{p'}$ for all $1 \le p < \infty$. Furthermore, the L^p norm of a function can be obtained via duality when $1 \le p \le \infty$ as follows:

$$\|f\|_{L^p} = \sup_{\|g\|_{L^{p'}}=1} \left| \int_X f \, g \, d\mu \right| .$$

For the endpoint cases $p = 1$, $p = \infty$, see Exercise 1.4.12(a), (b).

1.1.a. The Distribution Function

Definition 1.1.1. For f a measurable function on X, the *distribution function* of f is the function d_f defined on $[0,\infty)$ as follows:

$$d_f(\alpha) = \mu(\{x \in X : |f(x)| > \alpha\}). \tag{1.1.5}$$

The distribution function d_f provides information about the size of f but not about the behavior of f itself near any given point. For instance, a function on $\mathbf{R}^n$ and any of its translates have the same distribution function. It follows from Definition 1.1.1 that d_f is a decreasing function of α (not necessarily strictly).

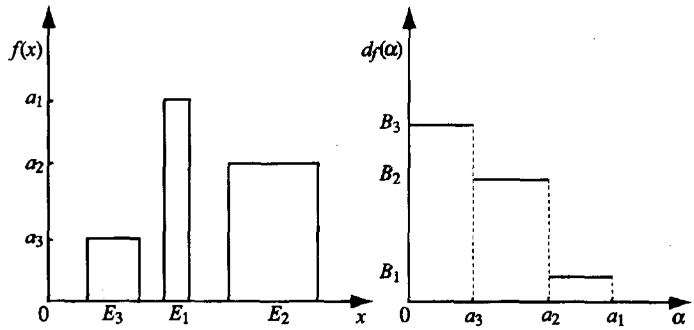

FIGURE 1.1. The graph of a simple function $f = \sum_{k=1}^{3} a_k \chi_{E_k}$ and its distribution function $d_f(\alpha)$. Here $B_j = \sum_{k=1}^{j} \mu(E_k)$.

Example 1.1.2. Recall that simple functions are finite linear combinations of characteristic functions of sets of finite measure. For pedagogical reasons we compute the distribution function d_f of a nonnegative simple function

$$f(x) = \sum_{j=1}^{N} a_j \chi_{E_j}(x),$$

where the sets E_j are pairwise disjoint and $a_1 > \cdots > a_N > 0$. If $\alpha \ge a_1$, then clearly $d_f(\alpha) = 0$. However, if $a_2 \le \alpha < a_1$ then $|f(x)| > \alpha$ precisely when $x \in E_1$ and, in general, if $a_{j+1} \le \alpha < a_j$, then $|f(x)| > \alpha$ precisely when $x \in E_1 \cup \cdots \cup E_j$. Setting

$$B_j = \sum_{k=1}^{j} \mu(E_k)$$

we have

$$d_f(\alpha) = \sum_{j=1}^{N} B_j \chi_{[a_{j+1}, a_j)}(\alpha),$$

where $a_{N+1} = 0$. Figure 1.1 illustrates this example when $N = 3$.

We now state a few simple facts about the distribution function d_f. We have

Proposition 1.1.3. *Let f and g be measurable functions on (X, μ). Then for all $\alpha, \beta > 0$ we have*