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John L. Kelley

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PREFACE

This book is a systematic exposition of the part of general topology which has proven useful in several branches of mathematics. It is especially intended as background for modern analysis, and I have, with difficulty, been prevented by my friends from labeling it: What Every Young Analyst Should Know.

The book, which is based on various lectures given at the University of Chicago in 1946-47, the University of California in 1948-49, and at Tulane University in 1950-51, is intended to be both a reference and a text. These objectives are somewhat inconsistent. In particular, as a reference work it offers a reasonably complete coverage of the area, and this has resulted in a more extended treatment than would normally be given in a course. There are many details which are arranged primarily for reference work; for example, I have taken some pains to include all of the most commonly used terminology, and these terms are listed in the index. On the other hand, because it is a text the exposition in the earlier chapters proceeds at a rather pedestrian pace. For the same reason there is a preliminary chapter, not a part of the systematic exposition, which covers those topics requisite to the main body of work that I have found to be new to many students. The more serious results of this chapter are theorems on set theory, of which a systematic exposition is given in the appendix. This appendix is entirely independent of the remainder of the book, but with this exception each part of the book presupposes all earlier developments.

There are a few novelties in the presentation. Occasionally the title of a section is preceded by an asterisk; this indicates that the section constitutes a digression. Other topics, many of equal or greater interest, have been treated in the problems. These problems are supposed to be an integral part of the discussion. A few of them are exercises which are intended simply to aid in understanding the concepts employed. Others are counter examples, marking out the boundaries of possible theorems. Some are small theories which are of interest in themselves, and still others are introductions to applications of general topology in various fields. These last always include references so that the interested reader (that elusive creature) may continue his reading. The bibliography includes most of the recent contributions which are pertinent, a few outstanding earlier contributions, and a few "cross-field" references.

I employ two special conventions. In some cases where mathematical content requires "if and only if" and euphony demands something less I use Halmos' "iff." The end of each proof is signalized by ■. This notation is also due to Halmos.

J. L. K.

Berkeley, California
February 1, 1955

ACKNOWLEDGMENTS

It is a pleasure to acknowledge my indebtedness to several colleagues.

The theorems surrounding the concept of even continuity in chapter 7 are the joint work of A. P. Morse and myself and are published here with his permission. Many of the pleasanter features of the appended development of set theory are taken from the unpublished system of Morse, and I am grateful for his permission to use these; he is not responsible for inaccuracies in my writing. I am also indebted to Alfred Tarski for several conversations on set theory and logic.

I owe thanks to several colleagues who have read part or all of the manuscript and made valuable criticisms. I am particularly obliged to Isaku Namioka, who has corrected a grievous number of errors and obscurities in the text and has suggested many improvements. Hugo Ribeiro and Paul R. Halmos have also helped a great deal with their advice.

Finally, I tender my very warm thanks to Tulane University and to the Office of Naval Research for support during the preparation of this manuscript. This book was written at Tulane University during the years 1950-52; it was revised in 1953, during tenure of a National Science Foundation Fellowship and a sabbatical leave from the University of California.

J. L. K.

April 21, 1961

A number of corrections have been made in this printing of the text. I am indebted to many colleagues, and especially to Krehe Ritter, for bringing errors to my attention.

J. L. K.

CONTENTS

CHAPTER 0: PRELIMINARIES

	PAGE
SETS	1
SUBSETS AND COMPLEMENTS; UNION AND INTERSECTION .	2
Class calculus	
RELATIONS	6
Relation calculus, equivalence relations	
FUNCTIONS	10
ORDERINGS	13
Order-complete sets, chains, extension of order-preserving functions	
ALGEBRAIC CONCEPTS	17
THE REAL NUMBERS	19
Integers, definition by induction, b -adic expansions	
COUNTABLE SETS	25
Subsets, unions, the set of real numbers	
CARDINAL NUMBERS	27
Schroeder-Bernstein theorem	
ORDINAL NUMBERS	29
The first uncountable ordinal	
CARTESIAN PRODUCTS	30
HAUSDORFF MAXIMAL PRINCIPLE	31
Maximal principle, Kuratowski-Zorn lemma, axiom of choice, well-ordering principle	

CHAPTER 1: TOPOLOGICAL SPACES

	PAGE
TOPOLOGIES AND NEIGHBORHOODS	37
Comparison of topologies, neighborhood system of a point	
CLOSED SETS	40
ACCUMULATION POINTS	40
CLOSURE	42
Kuratowski closure operators	
INTERIOR AND BOUNDARY	44
BASES AND SUBBASES	46
Topologies with a countable base, Lindelöf theorem	
RELATIVIZATION; SEPARATION	50
CONNECTED SETS	53
Components	
PROBLEMS	55
A Largest and smallest topologies; B Topologies from neighborhood systems; C Topologies from interior operators; D Accumulation points in T_1 -spaces; E Kuratowski closure and complement problem; F Exercise on spaces with a countable base; G Exercise on dense sets; H Accumulation points; I The order topology; J Properties of the real numbers; K Half-open interval space; L Half-open rectangle space; M Example (the ordinals) on 1st and 2nd countability; N Countable chain condition; O The Euclidean plane; P Example on components; Q Theorem on separated sets; R Finite chain theorem for connected sets; S Locally connected spaces; T The Brouwer reduction theorem	

CHAPTER 2: MOORE-SMITH CONVERGENCE

INTRODUCTION	62
DIRECTED SETS AND NETS	65
Uniqueness of limits, iterated limits	
SUBNETS AND CLUSTER POINTS	69
SEQUENCES AND SUBSEQUENCES	72
*CONVERGENCE CLASSES	73
Specification of a topology by convergence	
PROBLEMS	76
A Exercise on sequences; B Example: sequences are in-	

adequate; C Exercise on Hausdorff spaces: door spaces;
 D Exercise on subsequences; E Example: cofinal sub-
 sets are inadequate; F Monotone nets; G Integration
 theory, junior grade; H Integration theory, utility grade;
 I Maximal ideals in lattices; J Universal nets; K Bool-
 ean rings: there are enough homomorphisms; L Filters

CHAPTER 3: PRODUCT AND QUOTIENT SPACES

CONTINUOUS FUNCTIONS	84
Characterizations of continuity, homeomorphisms	
PRODUCT SPACES	88
Functions to a product, coordinatewise convergence, countability	
QUOTIENT SPACES	94
Open and closed maps, upper semi-continuous decom- positions	
PROBLEMS	100
A Connected spaces; B Theorem on continuity; C Ex- ercise on continuous functions; D Continuity at a point; continuous extension; E Exercise on real-valued con- tinuous functions; F Upper semi-continuous functions; G Exercise on topological equivalence; H Homeomor- phisms and one-to-one continuous maps; I Continuity in each of two variables; J Exercise on Euclidean n -space; K Exercise on closure, interior and boundary in prod- ucts; L Exercise on product spaces; M Product of spaces with countable bases; N Example on products and sep- arability; O Product of connected spaces; P Exercise on T_1 -spaces; Q Exercise on quotient spaces; R Exam- ple on quotient spaces and diagonal sequences; S Topo- logical groups; T Subgroups of a topological group; U Factor groups and homomorphisms; V Box spaces; W Functionals on real linear spaces; X Real linear topological spaces	

CHAPTER 4: EMBEDDING AND METRIZATION

EXISTENCE OF CONTINUOUS FUNCTIONS	112
Tychonoff lemma, Urysohn lemma	

CONTENTS		xi
		PAGE
EMBEDDING IN CUBES		115
Embedding lemma, Tychonoff spaces		
METRIC AND PSEUDO-METRIC SPACES		118
Metric topology, countable products		
METRIZATION		124
Urysohn metrization theorem, locally finite covers, refinement, characterization of metrizability		
PROBLEMS		130
A Regular spaces; B Continuity of functions on a metric space; C Problem on metrics; D Hausdorff metric for subsets; E Example (the ordinals) on the product of normal spaces; F Example (the Tychonoff plank) on subspaces of normal spaces; G Example: products of quotients and non-regular Hausdorff spaces; H Hereditary, productive, and divisible properties; I Half-open interval space; J The set of zeros of a real continuous function; K Perfectly normal spaces; L Characterization of completely regular spaces; M Upper semi-continuous decomposition of a normal space		

CHAPTER 5: COMPACT SPACES

EQUIVALENCES	135
Finite intersection property, cluster points, Alexander subbase theorem	
COMPACTNESS AND SEPARATION PROPERTIES	140
Compactness for Hausdorff, regular and completely regular spaces	
PRODUCTS OF COMPACT SPACES	143
The Tychonoff product theorem	
LOCALLY COMPACT SPACES	146
QUOTIENT SPACES	147
U.s.c. decompositions with compact members	
COMPACTIFICATION	149
Alexandroff one point and Stone-Čech compactifications	
LEBESGUE'S COVERING LEMMA	154
Even coverings	
*PARACOMPACTNESS	156

PROBLEMS	161
A Exercise on real functions on a compact space; B Compact subsets; C Compactness relative to the order topology; D Isometries of compact metric spaces; E Countably compact and sequentially compact spaces; F Compactness; the intersection of compact connected sets; G Problem on local compactness; H Nest characterization of compactness; I Complete accumulation points; J Example: unit square with dictionary order; K Example (the ordinals) on normality and products; L The transfinite line; M Example: the Helly space; N Examples on closed maps and local compactness; O Cantor spaces; P Characterization of Stone-Čech compactification; Q Example (the ordinals) on compactification; R The Wallman compactification; S Boolean rings: Stone representation theorem; T Compact connected spaces (the chain argument); U Fully normal spaces; V Point finite covers and metacompact spaces; W Partition of unity; X The between theorem for semi-continuous functions; Y Paracompact spaces	

CHAPTER 6: UNIFORM SPACES

UNIFORMITIES AND THE UNIFORM TOPOLOGY	175
Neighborhoods, bases and subbases	
UNIFORM CONTINUITY; PRODUCT UNIFORMITIES	180
Uniform isomorphism, relativization, products	
METRIZATION	184
Characterization of metrizability, the gauge of a uniformity	
COMPLETENESS	190
Cauchy nets, extension of functions	
COMPLETION	195
Existence and uniqueness	
COMPACT SPACES	197
Uniqueness of uniformity, total boundedness	
FOR METRIC SPACES ONLY	200
Baire theorem, localization of category, uniformly open maps	

	PAGE
PROBLEMS	203
A Exercise on closed relations; B Exercise on the product of two uniform spaces; C A discrete non-metrizable uniform space; D Exercise: uniform spaces with a nested base; E Example: a very incomplete space (the ordinals); F The subbase theorem for total boundedness; G Some extremal uniformities; H Uniform neighborhood systems; I Écarts and metrics; J Uniform covering systems; K Topologically complete spaces: metrizable spaces; L Topologically complete spaces: uniformizable spaces; M The discrete subspace argument; countable compactness; N Invariant metrics; O Topological groups: uniformities and metrization; P Almost open subsets of a topological group; Q Completion of topological groups; R Continuity and openness of homomorphisms: the closed graph theorem; S Summability; T Uniformly locally compact spaces; U The uniform boundedness theorem; V Boolean σ -rings	

CHAPTER 7: FUNCTION SPACES

POINTWISE CONVERGENCE	217
Topology and uniformity, compactness	
COMPACT OPEN TOPOLOGY AND JOINT CONTINUITY	221
Uniqueness of jointly continuous topologies, c.o. compact spaces	
UNIFORM CONVERGENCE	225
Uniform convergence on a family of sets, completeness	
UNIFORM CONVERGENCE ON COMPACTA	229
Topology, completeness, k -spaces	
COMPACTNESS AND EQUICONTINUITY	231
The Ascoli theorem	
*EVEN CONTINUITY	234
Topological Ascoli theorem	
PROBLEMS	238
A Exercise on the topology of pointwise convergence; B Exercise on convergence of functions; C Pointwise convergence on a dense subset; D The diagonal process	

and sequential compactness; E Dini's theorem; F Continuity of an induced map; G Uniform equicontinuity; H Exercise on the uniformity $\mathfrak{u}|_{\mathfrak{a}}$; I Continuity of evaluation; J Subspaces, products and quotients of k -spaces; K The k -extension of a topology; L Characterization of even continuity; M Continuous convergence; N The adjoint of a normed linear space; O Tietze extension theorem; P Density lemma for linear subspaces of $C(X)$; Q The square root lemma for Banach algebras; R The Stone-Weierstrass theorem; S Structure of $C(X)$; T Compactification of groups; almost periodic functions

APPENDIX: ELEMENTARY SET THEORY

CLASSIFICATION AXIOM SCHEME	251
Axiom of extent and classification axiom scheme	
CLASSIFICATION AXIOM SCHEME (<i>Continued</i>)	253
Formal statement of classification axiom scheme	
ELEMENTARY ALGEBRA OF CLASSES	253
EXISTENCE OF SETS	256
Axiom of subsets, axiom of union, unordered pairs	
ORDERED PAIRS; RELATIONS	259
FUNCTIONS	260
Axiom of substitution, axiom of amalgamation	
WELL ORDERING	262
Existence and uniqueness of order preserving functions	
ORDINALS	266
Axiom of regularity, structure of ordinals, transfinite induction	
INTEGERS	271
Axiom of infinity, Peano postulates for integers	
THE CHOICE AXIOM	272
The maximal principle	
CARDINAL NUMBERS	274
Elementary properties, finite sets, the product of cardinals	
BIBLIOGRAPHY	282
INDEX	293

Chapter 0

PRELIMINARIES

The only prerequisites for understanding this book are a knowledge of a few of the properties of the real numbers and a reasonable endowment of that invaluable quality, mathematical maturity. All of the definitions and basic theorems which are assumed later are collected in this first chapter. The treatment is reasonably self-contained, but, especially in the discussion of the number system, a good many details are omitted. The most profound results of the chapter are theorems of set theory, of which a systematic treatment is given in the appendix. Because the chapter is intended primarily for reference it is suggested that the reader review the first two sections and then turn to chapter one, using the remainder of the chapter if need arises. Many of the definitions are repeated when they first occur in the course of the work.

SETS

We shall be concerned with sets and with members of sets. "Set," "class," "family," "collection," and "aggregate" are synonymous,* and the symbol ϵ denotes membership. Thus $x \epsilon A$ if and only if x is a member (an element, a point) of A . Two sets are identical iff they have the same members, and equality is

* This statement is not strictly accurate. There are technical reasons, expounded in the appendix, for distinguishing between two different sorts of aggregates. The term "set" will be reserved for classes which are themselves members of classes. This distinction is of no great importance here; with a single non-trivial exception, each class which occurs in the discussion (prior to the appendix) is also a set.

always used to mean identity. Consequently, $A = B$ if and only if, for each x , $x \in A$ when and only when $x \in B$.

Sets will be formed by means of braces, so that $\{x: \dots (\text{proposition about } x) \dots\}$ is the set of all points x such that the proposition about x is correct. Schematically, $y \in \{x: \dots (\text{proposition about } x) \dots\}$ if and only if the corresponding proposition about y is correct. For example, if A is a set, then $y \in \{x: x \in A\}$ iff $y \in A$. Because sets having the same members are identical, $A = \{x: x \in A\}$, a pleasant if not astonishing fact. It is to be understood that in this scheme for constructing sets " x " is a dummy variable, in the sense that we may replace it by any other variable that does not occur in the proposition. Thus $\{x: x \in A\} = \{y: y \in A\}$, but $\{x: x \in A\} \neq \{A: A \in A\}$.

There is a very useful rule about the construction of sets in this fashion. If sets are constructed from two different propositions by the use of the convention above, and if the two propositions are logically equivalent, then the constructed sets are identical. The rule may be justified by showing that the constructed sets have the same members. For example, if A and B are sets, then $\{x: x \in A \text{ or } x \in B\} = \{x: x \in B \text{ or } x \in A\}$, because y belongs to the first iff $y \in A$ or $y \in B$, and this is the case iff $y \in B$ or $y \in A$, which is correct iff y is a member of the second set. All of the theorems of the next section are proved in precisely this way.

SUBSETS AND COMPLEMENTS; UNION AND INTERSECTION

If A and B are sets (or families, or collections), then A is a **subset** (**subfamily**, **subcollection**) of B if and only if each member of A is a member of B . In this case we also say that A is **contained in** B and that B **contains** A , and we write the following: $A \subset B$ and $B \supset A$. Thus $A \subset B$ iff for each x it is true that $x \in B$ whenever $x \in A$. The set A is a **proper subset** of B (A is properly contained in B and B properly contains A) iff $A \subset B$ and $A \neq B$. If A is a subset of B and B is a subset of C , then clearly A is a subset of C . If $A \subset B$ and $B \subset A$, then $A = B$, for in this case each member of A is a member of B and conversely.

The **union** (sum, logical sum, join) of the sets A and B , written $A \cup B$, is the set of all points which belong either to A or to B ; that is, $A \cup B = \{x: x \in A \text{ or } x \in B\}$. It is understood that "or" is used here (and always) in the non-exclusive sense, and that points which belong to both A and B also belong to $A \cup B$. The **intersection** (product, meet) of sets A and B , written $A \cap B$, is the set of all points which belong to both A and B ; that is, $A \cap B = \{x: x \in A \text{ and } x \in B\}$. The **void set** (empty set) is denoted 0 and is defined to be $\{x: x \neq x\}$. (Any proposition which is always false could be used here instead of $x \neq x$.) The void set is a subset of every set A because each member of 0 (there are none) belongs to A . The inclusions, $0 \subset A \cap B \subset A \subset A \cup B$, are valid for every pair of sets A and B . Two sets A and B are **disjoint**, or **non-intersecting**, iff $A \cap B = 0$; that is, no member of A is also a member of B . The sets A and B **intersect** iff there is a point which belongs to both, so that $A \cap B \neq 0$. If \mathcal{A} is a family of sets (the members of \mathcal{A} are sets), then \mathcal{A} is a **disjoint family** iff no two members of \mathcal{A} intersect.

The **absolute complement** of a set A , written $\sim A$, is $\{x: x \notin A\}$. The **relative complement** of A with respect to a set X is $X \cap \sim A$, or simply $X \sim A$. This set is also called the **difference** of X and A . For each set A it is true that $\sim \sim A = A$; the corresponding statement for relative complements is slightly more complicated and is given as part of 0.2.

One must distinguish very carefully between "member" and "subset." The set whose only member is x is called **singleton** x and is denoted $\{x\}$. Observe that $\{0\}$ is not void, since $0 \in \{0\}$, and hence $0 \neq \{0\}$. In general, $x \in A$ if and only if $\{x\} \subset A$.

The two following theorems, of which we prove only a part, state some of the most commonly used relationships between the various definitions given above. These are basic facts and will frequently be used without explicit reference.

1 THEOREM *Let A and B be subsets of a set X . Then $A \subset B$ if and only if any one of the following conditions holds:*

$$A \cap B = A, \quad B = A \cup B, \quad X \sim B \subset X \sim A,$$

$$A \cap X \sim B = 0, \quad \text{or} \quad (X \sim A) \cup B = X.$$