Sterling K. Berberian

FUNDAMENTALS OF REAL ANALYSIS

实分析基础

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Fundamentals of Real Analysis

With 31 Figures

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To the memory of James Ellis Powell,
late Professor Emeritus, Michigan State University

Preface

This book is a record of a course on functions of a real variable, addressed to first-year graduate students in mathematics, offered in the academic year 1985–86 at the University of Texas at Austin. It consists essentially of the day-by-day lecture notes that I prepared for the course, padded up with the exercises that I seemed never to have the time to prepare in advance; the structure and contents of the course are preserved faithfully, with minor cosmetic changes here and there.

Two facts are worth noting: (1) the lecture notes were prepared (if not always delivered) with exceptional care, as my son was enrolled in the class and I confess that I was trying especially hard to put my best foot forward; (2) the text does not reflect the fact that I wasted a certain amount of time doing Lebesgue's "Fundamental theorem of calculus" at the end of the first semester, 'discovered' E.J. McShane's lovely exposition during the semester break, and was so struck by the superiority of his exposition that I did the topic all over again at the beginning of the second semester. It is only the 'second pass' that is recorded here (in Chapter 5); the time saved by doing it right in the first place should be ample for including the very few topics I added that were not covered in the actual course (notably, the Riesz representation theorem, included here as Theorem 6.7.11—the 11th item in §7 of Chapter 6).

The choice of topics and the order in which they are taken up was guided by the following principles:

(1) The most important things should come first (it is a little intellectually arrogant to make such judgments, but that's what a teacher is paid to do—and the student need not, and sometimes should not, agree). When planning the course, at each topic I kept in mind the question: "If the student is obliged to drop out tomorrow—or who takes only the first semester, as is frequently the case—will he or she have been exposed to the topics that are most likely to be crucial in his or her mathematical development?"

(2) Every subject becomes fatiguing after a while, and when fatigue sets in, learning converges rapidly to zero. For example, the course syllabus called for a full-dress treatment of measure and integration, but consuming it all in one gulp leads to indigestion (I ask forgiveness of all the students on whom I inflicted one-semester or even one-year courses in Measure and

Integration; we got some good out of it and I amassed enough material for a book on the subject, but it was not the best use of our time). Therefore, the theme of measure theory must be broken up into digestible units and alternated with other themes for the sake of variety. The same is true of topology and function spaces: a generous portion, but not all in one gulp.

(3) The house being built, to be sturdy and serviceable, must have a foundation: the first part of the course must come to grips with the real numbers (they have to be constructed rigorously from the rationals), the axioms of set theory (just visiting!) and the concepts of cardinality and ordinality (indispensable tools in grappling with infinity, one of the mathematician's principal occupations); for an eloquent essay on the importance of taking up such matters, I refer the reader to the Preface of Irving Kaplansky's *Set theory and metric spaces* [2nd edn., Chelsea, New York, 1977].

A certain amount of inefficiency is introduced in the passage from concrete to abstract (measure spaces), special to general (metric and topological spaces), finite to infinite (product measure, signed measures), real to complex (function spaces), and so on. This seemed not burdensome in the classroom, where a few words often sufficed to reset the stage for the reappearance of a subject, but in print it is necessary to revisit a considerable amount of notation and definitions, especially when related discussions are widely separated in time (pages). The benefits of recurrent themes (motivation, boredom avoidance) seemed worth the inefficiency in class; I hope the reader will find that they also make the book easier to read.

Can the topics taken up be treated more effectively? Assuredly. Could I have chosen more important topics to take up? At the time, I thought not, and, a decade later, I feel sufficiently comfortable with the choices to warrant putting the lecture notes into a more presentable form; the ultimate verdict, as always, is the reader's.

Austin, Texas
September 1996

Sterling K. Berberian

Contents

CHAPTER 1

Foundations

The reader will already have a working familiarity with the concepts of set and function. Apart from a review of basic concepts and notations, the chapter is mostly about coping with infinity (or exploiting it—it depends on one's point of view). Our viewpoint (generally called 'naive') is that infinite sets exist and they are not to be feared. Some of the axioms of set theory (the Axiom of Choice, the Continuum Hypothesis) are more controversial than the others; whether or not one admits them is a matter of professional lifestyle.[1] In this text, the axiom of choice and its logically equivalent forms (§1.11) are admitted and are invoked whenever convenient. The continuum hypothesis is only mentioned briefly (§1.13); although it is not used anywhere in the text, it is instructive to understand the terms needed to state it.

[1] Assuming the usual (Zermelo-Fraenkel) axioms of set theory are consistent [cf. I. Kaplansky, *Set theory and metric spaces*, Chapter 3, 2nd edn., Chelsea, New York, 1977]. If ZF goes down the tube, we all go down with it.

1.1. Logic, Set Notations[1]

1.1.1. A 'short list' of useful symbols:

Notation	*Read*
$x \in A$	x is an element of the set A
$\forall$	for all, for every
$\exists$	there exists (at least one)
$\exists!$	there exists a unique (one and only one)
$\ni$	such that (having the following properties)
&, $\wedge$	and
$\vee$	or (non-exclusive)
$\Rightarrow$	implies
$\Leftrightarrow$	if and only if (a fusion of $\Rightarrow$ and $\Leftarrow$)
$\sim$	negation (of a proposition)

1.1.2. A *proposition* is a statement that is either true or false (but not both). If P is a proposition, its *negation* $\sim$ P is the proposition that is false when P is true, true when P is false. For example, in ordinary arithmetic, if P is the (false) proposition $\ll 3 \leq 2 \gg$ then $\sim$ P is the (true) proposition $\ll 2 < 3 \gg$; more generally, if P is $\ll x \leq y \gg$ then $\sim$ P is $\ll y < x \gg$.

1.1.3. Before explaining the usage of the other symbols, it helps to have a repertory of specific sets:

Symbol	*Meaning*
$\mathbb{P}$	the set $\{1, 2, 3, \ldots\}$ of all *positive integers*
$\mathbb{N}$	the set $\{0, 1, 2, 3, \ldots\}$ of all *nonnegative integers*
$\mathbb{Z}$	the set $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$ of all *integers*
$\mathbb{Q}$	the set of all *rational numbers* m/n $(m, n \in \mathbb{Z};\ n \neq 0)$
$\mathbb{R}$	the set of all *real numbers*
$\mathbb{C}$	the set of all *complex numbers* $z = x + iy$ $(x, y \in \mathbb{R},\ i^2 = -1)$

The construction of $\mathbb{R}$ from the rational field $\mathbb{Q}$ is sketched in §1.8; the construction of $\mathbb{C}$ from $\mathbb{R}$ is elementary algebra. The notations $\mathbb{R}$ and $\mathbb{C}$ are standard; $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are 'fairly standard' (i.e., widely used); $\mathbb{P}$ is improvised (no consensus!).

1.1.4. If P and Q are propositions, then P $\Rightarrow$ Q means that if P is true then Q is also true. For example, the statement

$$x \in \mathbb{Z} \ \Rightarrow \ x \in \mathbb{Q}$$

[1] I suggest first glancing at the tables in this section; if everything looks familiar, the section can be omitted.

says that every integer is a rational number; it is true. The *converse* statement (with implication pointing in the reverse direction)

$$x \in \mathbb{Z} \Leftarrow x \in \mathbb{Q}$$

happens to be false, but it is a legitimate statement. (The rules of ordinary language do not abolish lies.) When we demonstrate that $P \Rightarrow Q$, we say that we have proved a *theorem*, with *hypothesis* P and *conclusion* Q. Sometimes (often!) a theorem $P \Rightarrow Q$ is proved by showing that $\sim Q \Rightarrow \sim P$ (the *contrapositive* form of $P \Rightarrow Q$).

1.1.5. If propositions P and Q imply each other, they are said to be logically *equivalent*, written $P \Leftrightarrow Q$ (or $P \equiv Q$); thus

$$(P \Leftrightarrow Q) \equiv (P \Rightarrow Q) \ \& \ (Q \Rightarrow P) .$$

For example, in ordinary arithmetic,

$$x = y \Leftrightarrow (x \le y) \ \& \ (y \le x) .$$

The basis of proofs in contrapositive form is the equivalence

$$(P \Rightarrow Q) \equiv (\sim Q \Rightarrow \sim P) .$$

An equivalence of some depth: for a real number x,

$$x \geq 0 \Leftrightarrow (\exists \, y \in \mathbb{R} \ni x = y^2)$$

(translate it from 'symbolese' into ordinary language!).

1.1.6. The symbol $\forall$ is sometimes used literally, sometimes as a 'stage-setter' (or 'quantifier') indicating the set in which a statement is formulated. Consider, for example, the statements

$$x^2 \geq 0 \quad (\forall \, x \in \mathbb{R})$$

$$(\forall \, x \in \{1,2,3,4\}) \ x = 1 \Leftrightarrow x^2 < 4 .$$

In the first example, the condition on x ($x^2 \geq 0$) is true for every x in the set $\mathbb{R}$, thus the statement simply says that x^2 is nonnegative for *every* real number x; here, $\forall$ is used in its ordinary, literal sense. In the second example, the condition on x (the assertion of an equivalence $\Leftrightarrow$) is also true for every x in the set $\{1,2,3,4\}$ (albeit in a vacuous way for $x = 2$, 3 or 4), though its constituent pieces ($x = 1$, $x^2 < 4$) are not.

1.1.7. The mathematical 'or' is used 'permissively' rather than 'exclusively'; thus, the statement

$$(x \in A) \vee (x \in B)$$

does not exclude the possibility that both $x \in A$ and $x \in B$.

1.1.8. If A and B are sets such that $x \in A \Rightarrow x \in B$, then A is called a *subset* of B, written $A \subset B$ (alternatively, B is a *superset* of A, written $B \supset A$). For example, the set E of even integers is a subset of $\mathbb{Z}$; it can be specified as the set of all integers n such that $n = 2k$ for some integer k, a recipe conveniently expressed by

$$E = \{n \in \mathbb{Z} : \ n = 2k \text{ for some } k \in \mathbb{Z}\}$$

(the colon is read as "such that"). More generally, if X is a set and if, for each $x \in X$, $P(x)$ is a proposition involving x, then

$$\{x \in X : \ P(x)\}$$

denotes the set of all elements x of X for which $P(x)$ is true; this can be shortened to $\{x : \ P(x)\}$ when there is no doubt as to the 'universal set' X from which the elements x are drawn. For example,

$$\{x \in \mathbb{Z} : \ -2 \le x < 4\} = \{-2, -1, 0, 1, 2, 3\}$$

(a set with six elements), whereas

$$\{x \in \mathbb{R} : \ -2 \le x < 4\} = [-2, 4)$$

(a semi-closed interval); unless a universal set (such as $\mathbb{Z}$ or $\mathbb{R}$) is specified, the notation $\{x : \ -2 \le x < 4\}$ is ambiguous. An expression such as $\{x : \ x \in A \ \& \ x \in B\}$ is unambiguous, since it can be rewritten as $\{x \in A : \ x \in B\}$.

1.1.9. Fix a universal set X and let A, B, C, ... be subsets of X. The following table lists the most basic set-theoretic notations (others follow in later sections):

Symbol	*Meaning*
$x \notin A$	$\sim (x \in A)$ (that is, x *not* an element of A)
$A \subset B$	$x \in A \Rightarrow x \in B$
$A \not\subset B$	$\sim (A \subset B)$ (that is, $\exists\, x \in A \ni x \notin B$)
$A = B$	$x \in A \Leftrightarrow x \in B$ (that is, $A \subset B$ and $B \subset A$)
$A \ne B$	$\sim (A = B)$ (that is, either $A \not\subset B$ or $B \not\subset A$)
$A \subsetneqq B$	$A \subset B \ \& \ A \ne B$ (that is, $A \subset B$ and $B \not\subset A$; A is then said to be a *proper* subset of B, and B is said to contain A *properly*)
$B \supset A$	$A \subset B$
$A \cap B$	$\{x : \ x \in A \ \& \ x \in B\}$ (the *intersection* of A and B)
$A \cup B$	$\{x : \ x \in A \text{ or } x \in B\}$ (the *union* of A and B)
$\complement_X A$	$\{x \in X : \ x \notin A\}$ (the *complement* of A in X)
$A - B$	$\{x : \ x \in A \ \& \ x \notin B\}$ (the *difference* 'A minus B', or the 'relative complement' of B in A)

The complement of A in X is also written $\complement A$ or A'; thus $A - B = A \cap \complement B = A \cap B'$, $(A')' = A$, $A \cup A' = X$ and $A \cap A' = \emptyset$ (the *empty* *set*). Some other useful formulas are listed in the exercises for convenient reference.

Finally, a caution about the use of the words 'all' and 'every':

1.1.10. (*Russell's paradox*)[2] The statement "There exists a set of which every set is a member" is nonsense. For, if U were such a set, then its subset

$$A = \{x \in U : x \notin x\}$$

would be a member of U, leading inexorably to a contradiction: either $A \in A$ (in which case $A \notin A$ by the definition of A), or $A \notin A$ (in which case $A \in A$ by the definition of A).

Moral. The words 'all' and 'every' are very big (too big); to play it safe, qualify by operating within a known set. The following usage of 'all' is prudent: 'The set of all one-element subsets of $\mathbb{P}$'; the sets A in question are qualified by the condition $A \subset \mathbb{P}$. The expression 'The set of all one-element sets' is asking for trouble. {Trouble: Let E be 'the set of all one-element sets', then consider the set F of all sets A that contain an element of E (in other words, $A \neq \emptyset$); we are now face to face with $F \cup \{\emptyset\}$, the dreaded 'set of all sets'.}

Exercises

1. Let X be a set, A, B, C subsets of X, A' the complement of A.
(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(i′) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(ii) $A \subset B \Leftrightarrow A' \supset B'$
(iii) $(A \cup B)' = A' \cap B'$
(iii′) $(A \cap B)' = A' \cup B'$
(iv) $A \subset B \Leftrightarrow A = A \cap B$
(iv′) $A \subset B \Leftrightarrow B = A \cup B$

2. The description of a "proposition" in 1.1.2 can be expressed as follows: For every proposition P, $P \vee (\sim P)$ is true (*law of the excluded middle*) and $P \wedge (\sim P)$ is false (*law of contradiction*).

1.2. Relations

1.2.1. *Definition.* If X and Y are sets, the **cartesian product** of X and Y (in that order), denoted $X \times Y$, is the set of all ordered pairs

---

[2] Bertrand Russell (1872-1970).

(x, y) with $x \in X$ and $y \in Y$:

$$X \times Y = \{(x, y) : x \in X \,\&\, y \in Y\},$$

with the understanding that

$$(x, y) = (x', y') \Leftrightarrow x = x' \,\&\, y = y'.$$

One calls x and y the first and second *coordinates* of (x, y) (cf. Figure 1).

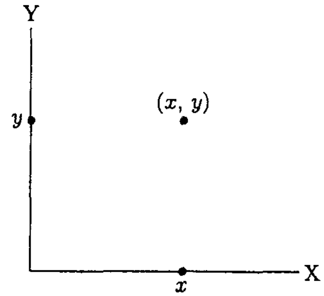

Figure 1

1.2.2. *Definition.* A **relation** from X to Y (in that order) is a subset R of $X \times Y$:

$$R \subset X \times Y$$

(cf. Figure 2). If $(x, y) \in R$ we write xRy (read "x is related by R to y"), and if $(x, y) \notin R$ we write $xR'y$ (an appropriate notation, since (x, y) belongs to the complement R' of R). If $X = Y$ we say that R is a relation *in* X.

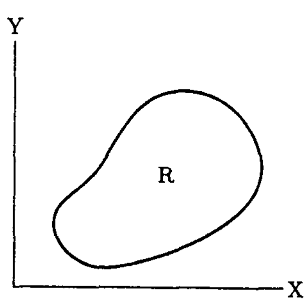

Figure 2

1.2.3. *Example.* Let $X = \{1,2,3,4\}$ and let R be the usual relation "<" in X; as a set of ordered pairs,

$$R = \{(1,2),\ (1,3),\ (1,4),\ (2,3),\ (2,4),\ (3,4)\}.$$

1.2.4. *Example.* Let $X = \{1,2,3,4\}$ and suppose that xRy means that $x|y$ (x is a divisor of y). Then

$$R = \{(1,1),\ (1,2),\ (1,3),\ (1,4),\ (2,2),\ (2,4),\ (3,3),\ (4,4)\}.$$

1.2.5. *Example.* If R is a relation in X and A is a subset of X, then $R \cap (A \times A)$ is a relation in A, said to be **induced** in A by R.

1.2.6. *Definition.* Let R be a relation from X to Y (1.2.2). For each subset A of X, we write

$$R(A) = \{y \in Y : xRy \text{ for some } x \in A\}$$

and call it the (direct) **image** of A under R; for each subset B of Y, we write

$$R^{-1}(B) = \{x \in X : xRy \text{ for some } y \in B\}$$

and call it the **inverse image** of B under R (cf. Figure 3).

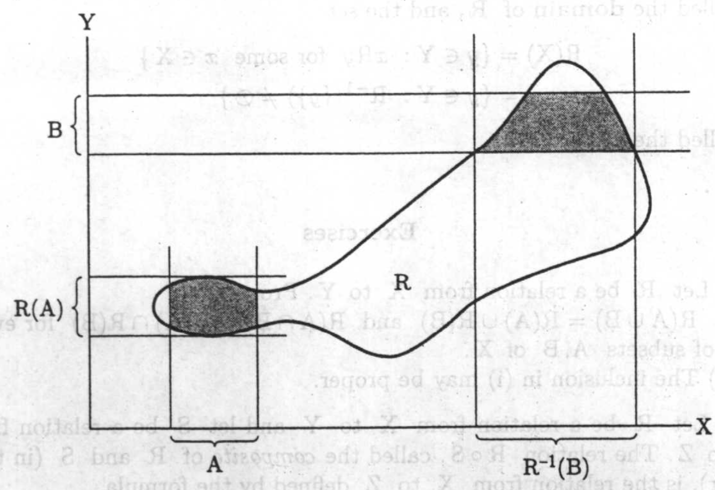

Figure 3

1.2.7. *Remarks.* With notations as in 1.2.6, one can think of R^{-1} as a relation from Y to X, where

$$yR^{-1}x \Leftrightarrow xRy,$$

that is,

$$R^{-1} = \{(y,x) : (x,y) \in R\};$$