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Vojislav Marić

Regular Variation and Differential Equations

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Preface

The notion of regular variation was discovered by Jovan Karamata in his famous paper of 1930 "Sur une mode des croissance regulière des fonctions". Karamata's aim was Tauberian theory, one of the highlights of the epoch marked by the work of eminent analysts, predominantly that of G.H. Hardy, J.L. Littlewood and also of E. Landau, culminating in N. Wiener's general Tauberian theorem in 1932. However, in addition to proving Tauberian theorems first for Laplace-Stieltjes and later for the more general integral transforms of convolution type, regular variation was soon applied in Abelian theorems, giving in fact asymptotic behavior of integrals and series, the Fourier ones in particular. Further applications in analysis include Mercerian theorems, analytic number theory, complex analysis-entire functions in particular. With W. Feller's well known treatise of 1968, [14], regular variation was recognized as a major tool in the probability theory and its applications. A new impetus to the subject was provided by the L. de Haan work in 1970, [23], where he introduced a substantial generalization of regular variation, aiming again primarily at the probability theory. This can be found in the monograph of J.L. Geluk and L. de Haan of 1987, [18].

The first paper connecting regular variation and the differential equation is the one of V.G. Avakumović of 1947, "Sur l'équation différentielle de Thomas-Fermi". His paper did not attract much attention - regularly varying functions were totally distant from the theory of differential equation at that time, until the investigations of M. Tomić and the author started in 1976, [37]. The first study of the linear equations is that of E. Omey in 1981, [55].

The most complete presentation of Karamata theory and its generalizations as well as the majority of the applications are contained in the book of 1987 by N.H. Bingham, C.M. Goldie and J.L. Teugels [9]. Rudimentary results on differential equations form its Appendix 2.

The first monograph on the subject is the one of E. Seneta of 1976, [60].

The core of this treatise is based on joint results of Miodrag Tomić and the author. Significant contribution to the main theme of the book are the included results of J.L. Geluk and E. Omey and the joint ones of H.C. Howard and the author. Although Miodrag Tomić is formally not an author of this book, the whole text is permeated with his influence and ideas. This holds both for conjectures leading to a number of theorems and for many special techniques and devices needed for the proofs where general methods fail. A long lasting cooperation with him has been for the author the most enlightening and inspiring experience in mathematics.

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Contents

Introduction

0.1. This book deals with some properties of solutions of the second order differential equation of the form

$$y'' = f(x)\phi(y) \tag{0.1}$$

both for the linear and for some nonlinear cases. The former one is extended to the more general equation $y''+g(x)y'+h(x)y=0$. The latter one contains some classes of equations modelling certain physical phenomena, and the natural generalizations.

Another equation studied here is

$$y''' - yy'' + \lambda(1+y'^2) = 0$$

arising in boundary layer theory.

In particular we are interested in the precise asymptotic behaviour of solutions y for the large value of the variable. By that we mean as usual: To find an ultimately continuous, positive function g such that $y(x)/g(x) \to 1$, as $x \to \infty$.

Concerning the linear equation the problem of asymptotics is very old and goes back to Liouville, [36], and Green, [20]. (See [27] for an historical survey). There is a voluminous literature on the subject for the more general case of linear systems, [13], [11]. It is also of contempory interest in theoretical physics since the considered linear equation represents the one-dimensional Schrödinger equation [61].

The novelty of the approach in this treatise consists of introducing into the study of properties - asymptotics in particular, of solutions of differential equations the notion of regular variation of Karamata as defined in Definitions 0.1 and 0.2 below and also of its extensions, in particular that of de Haan as given in Definitions 0.8 and 0.9.

For the linear equation (0.1) i.e. when $\phi(y) = y$, with ϕ of arbitrary sign, the necessary and sufficient conditions for the solutions to belong to Karamata class of functions (see paragraph 0.2), are formulated. (Ch. 1). This gives a new insight into the set of solutions since in the extensively developed theory (see e.g. [60], [9], [18]), numerous interesting and specific properties of functions of Karamata class are proved which are also suitable for various applications.

Conversely, this offers a differential characterization of a subset of Karamata functions.

It is shown in this book that several of the mentioned properties are in particular useful in obtaining a method leading to the asymptotic behaviour of solutions for both linear (Ch. 2.) and for the aforesaid classes of non-linear equations (Ch. 3., Ch. 4.). The asymptotic formulas for solutions obtained in this way contain as a factor a slowly varying function not necessarily tending to a constant and therefore differ from all previously known in the asymptotics of solutions of differential equations and also generalize the relevant earlier ones.

0.2. In order to be able to formulate our results we present here the definition of regularly (in particular, slowly) varying functions as introduced by J. Karamata, [33] and the two fundamental properties of these. In addition the complementary definition of rapidly varying functions, [5], the one of regularly bounded functions, [3]and of de Haan class [23] are given, to serve the same purpose as the previous ones.

Additional properties of Karamata and of related functions needed in performing the proofs, are presented in the Appendix.

Definition 0.1. *A positive measurable function ρ defined on some neighbourhood $[a,\infty)$ of infinity is called regularly varying at infinity of index α if, for each $\lambda > 0$ and some $\alpha \in \mathbb{R}$,*

$$\lim_{x\to\infty} \rho(\lambda x)/\rho(x) = \lambda^{\alpha}. \tag{0.2}$$

The real number α is called the index of regular variation.

Definition 0.2. *A positive measurable function L defined on some neighbourhood $[a,\infty)$ of infinity is called slowly varying at infinity, if for each $\lambda > 0$,*

$$\lim_{x\to\infty} L(\lambda x)/L(x) = 1. \tag{0.3}$$

It follows from the previous two definitions that if ρ is regularly varying of index α it can be represented in the form

$$\rho(x) = x^{\alpha} L(x).$$

It is also clear that a slowly varying function L is regularly varying of index $\alpha = 0$. Consequently, the set of slowly varying functions forms a subset of the set of regularly varying ones.

This is a somewhat misleading statement, since the class of slowly varying functions is the one which presents itself, due to wealth of interesting properties, as a major novelty in the classical analysis and its applications.

In the sequel the term "regularly varying function" sometimes will include the slowly regular ones and sometimes not. The context, however, will prevent any ambiguity.

We present the two fundamental properties which are the main source of most additional ones.

Theorem 0.1., [33] (Uniform convergence theorem). *If ρ is regularly varying of index $\alpha \in \mathbb{R}$ at infinity then the relation (0.2) (and so (0.3)) holds uniformly for $\lambda \in [a,b]$ with $0 < a < b < \infty$.*

Theorem 0.2., [33] (Representation theorem). *The function L is slowly varying at infinity if and only if it may be written in the form*

$$L(x) = c(x) \exp\left\{\int_a^x (\varepsilon(t)/t)dt\right\}, \quad x \geq a, \tag{0.4}$$

for some $a > 0$, where ε and c are measurable and for $x \to \infty$, $\varepsilon(x) \to 0$ and $c(x) \to c$, with $c \in (0,\infty)$.

(Notice that L, c, ε may be altered at will on finite intervals so that the value of a is unimportant and if $a = 0$ one can take $\varepsilon(x) \equiv 0$ in a neighbourhood of zero to avoid divergence of the integral at the origin.)

We emphasize at this point, that from some point of view, e.g. the measuring of scales of growth like in studying asymptotic behaviour of relevant functions, slowly varying functions are of interest only to within asymptotic equivalence. For that purpose it suffices to take in (0.4) $c(x)$ to be equal to a positive constant c.

The following definition of E.E. Kohlbecker is pertinent to the case.

Definition 0.3., [35]. *The slowly varying function*

$$L(x) = c\exp\left\{\int_a^x (\varepsilon(t)/t)dt\right\} \tag{0.5}$$

where c is a positive constant is called normalized.

That class will play an important role in the sequel.

Some examples of slowly varying functions are provided by:

$$L(x) = \prod_{\nu=1}^{n} (\log_\nu x)^{\xi_\nu}$$

where ξ_ν are real numbers and $\log_\nu$ denotes the ν-th iteration of the logarithm,

$$L(x) = \exp\left\{ \prod_{\nu=1}^{n} (\log_\nu x)^{\eta_\nu} \right\}$$

where $0 < \eta_\nu < 1$,

$$L(x) = \frac{1}{x}\int_a^x \frac{dt}{lnt}.$$

The above given examples of slowly varying functions might associate these with the monotonicity for large values of x. This is far from being true as indicates the following example

$$L(x) = \exp\left\{ (lnx)^{1/3} \cos(lnx)^{1/3} \right\}, \tag{0.6}$$

$$\text{where} \quad \liminf_{x\to\infty} L(x) = 0, \qquad \limsup_{x\to\infty} L(x) = \infty,$$

so that $L(x)$ oscillates infinitely.

The natural extension of the class of functions introduced by Definitions 0.1 and 0.2 was made only in 1957 by A. Békéssy [5] as follows:

Definition 0.4. *A positive measurable function g defined on some neighbourhood $[a, \infty)$ of infinity is called rapidly varying at infinity of index ∞ if for $x \to \infty$,*

$$g(\lambda x)/g(x) \to \begin{cases} \infty & , \; for\, \lambda > 1 \\ 0 & , \; for\, 0 < \lambda < 1 \end{cases} \tag{0.7)a}$$

and is called rapidly varying *at infinity of index $-\infty$ if for $x \to \infty$*

$$g(\lambda x)/g(x) \to \begin{cases} 0 & , \; for\, \lambda > 1 \\ \infty & , \; for\, 0 < \lambda < 1. \end{cases} \tag{0.7)b}$$

Together they are called rapidly varying at infinity.

For instance, $g(x) = e^x$ is rapidly varying of index ∞ and $g(x) = e^{-x}$ is such of index $-\infty$.

It is sometimes necessary to transfer attention from infinity to the origin as it is the case throughout the Chapter 4 of this book. We present, therefore

Definition 0.5. *A positive measurable function* ρ *defined on some neighbourhood* $(0,a)$, $a>0$ *is called regularly varying at zero of index* α, *if for each* $\lambda>0$ *and some* $\alpha\in\mathbb{R}$

$$\lim_{x\to 0^+}\rho(\lambda x)/\rho(x)=\lambda^{\alpha}. \tag{0.8}$$

Notice that this is equivalent to saying that $f(1/x)$ is regularly varying at ∞ of index $-\alpha$. Consequently, one can transfer properties of relevant functions from the case $x\to\infty$ to the case $x\to 0^+$.

We shall call the functions introduced by Definition 1-4, *Karamata class of functions.*

Some simple oscillating functions (e.g. $g(x)=2+\sin x$) are not regularly varying. A natural and useful generalization was given in 1935 by V. G. Avakumović [3] (cf. J. Karamata [34]). An alternative approach is given by N. K. Bari and S. B. Stečkin [6].

Definition 0.6. *A positive measurable function* g *defined on some neighbourhood* $[a,\infty)$ *of infinity is called regularly bounded (or R-O varying) at infinity if for each* $1\le\lambda\le\lambda_0$.

$$m\le g(\lambda x)/g(x)\le M$$

where, λ_0, m, M *are any constants satisfying* $1<\lambda_0<\infty$, $0<m<1$, $1<M<\infty$.

One can show that all regularly varying functions belong to this class. This is also true for all positive measurable functions which are on $[a,\infty)$ bounded away from both 0 and ∞.

In the context of regularly bounded functions the following notion will be needed:

Definition 0.7. *A function* $g(x)$ *is said to be almost increasing if there exists a constant* $A>1$ *such that* $x_2<x_1$ *implies* $g(x_2)\le Ag(x_1)$; *almost decreasing functions are defined likewise.*

0.3. In Definition 0.1 the asymptotic relation of the form $\phi(xt)/\phi(t)\to$

$\psi(x)$, as $t \to \infty$ is considered. Writting $f = ln\phi$, $k = ln\psi$ this becomes

$$f(xt) - f(t) \to k(x), \quad \text{as} \quad t \to \infty.$$

In [23] the Haan studied more general relation

$$\frac{f(tx) - f(t)}{a(t)} \to g(x) \quad \text{as} \quad t \to \infty, \quad \text{for all} \quad x > 0. \tag{0.9}$$

The function a is called the auxiliary function.

By that a new class of functions - de Haan class is introduced. An extensive theory of such functions rich in ramifications is developed, parallel to the one of Karamata class [18], [9, 3]. De Haan class present itself as very fruitful in various applications. Here we present some of these related to differential equations.

If in particular one put in (0.9) $g(x) = lnx$ a subclass of slowly varying functions can be obtained by

Definition 0.8., [18, Def. 1.11]. *A measurable function f defined on $[a_0, \infty)$ is said to belong to the class Π if there exists a positive function a defined on $(0, \infty)$ such that for $x > 0$*

$$\lim_{t \to \infty} \frac{f(xt) - f(t)}{a(t)} = lnx;$$

one writes $f \in \prod$ or $f \in \prod(a)$.

Furthermore, using Definition 0.8, one can introduce an useful subclass of rapidly varying functions as framed in

Definition 0.9., [18, Def. 1.24] *A positive non-decreasing function f defined on $\mathbb{R}$, is said to belong to the class Γ if there exist a positive function b defined on $\mathbb{R}$ such that for all $x \in \mathbb{R}$ and for $t \to \infty$*

$$\lim_{t \to \infty} \frac{f(t + xb(t))}{f(t)} = e^x,$$

the function b is called the auxiliary function for f.

One writes $f \in \Gamma$ or $f \in \Gamma(b)$.

We shall also make use of the following

Definition 0.10., [18, Def. 1.33] *A positive measurable function b defined on $\mathbb{R}$ is said to be Beurling slowly varying if for all $x \in \mathbb{R}$ and*

$$\lim_{t\to\infty} \frac{b(t+xb(t))}{b(t)} = 1;$$

one writes $b \in BSV$.

The above relation holds locally uniformly in x if b is continuous, [18, Th. 1.34].

0.4. One defines a *logarithmico-exponential function* as a real-valued function defined on some half-axis (a, ∞) by a finite combination of the symbols $+$, $-$, $\times$, $:$, $\sqrt[n]{\ }$, ln, exp, acting on the real variable x and on real constant, [25, 3.2].

More generally, one defines *Hardy field* as a set of germs of real-valued functions defined on some half-axis (a, ∞) that is closed under differentiation and that form a field under the usual addition and multiplication of germs, [57, p.297].

The logarithmico-exponential class of functions or more generally - Hardy fields, have been considered as the natural domain of asymptotic analysis where all rules hold without qualifying conditions.

In G. H. Hardy's own words [25, 4.5]: "No function has yet presented itself in analysis the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmico-exponential terms".

This statement of Hardy was basically influenced by the fact that the arithmetic functions occuring in the number theory having often very complicate structure and for which he expected "would give rise to genuinely new modes of increase", so far obey the log-exp laws of increase.

That indicates a possible significance of the results in this treatise as sketched in paragraph 1. For, any logarithmico-exponential function f (or any element of Hardy fields) together with the derivatives, is ultimately continuous and monotonic, of constant sign and $\lim_{x\to\infty} f(x)$ exists as a finite or infinite one. On the other hand a slowly varying function may oscillate - even infinitely as shown by the last preceeding example (0.6). And, as it is pointed out in paragraph 0.1, the solutions of a second order linear equation may behave as slowly varying functions. Therefore, the solutions of such a simple equation may exibit a "genuinely new mode of increase". To support our point we emphasize here that no hypothesis of the theorems which follow leading to the above statement concerning solutions, is related to regular variation (Cf. 1.2).

In addition we recall that the solutions of considered linear equation $y'' + f(x)y = 0$ with $f(x)$ of arbitrary sign can be oscillatory or nonoscillatory depending on properties of f. Since regularly (and slowly) varying functions are, by Definitions 0.1 and 0.2, ultimately positive, the necessary and sufficient conditions, mentioned in paragraph 0.1 ensuring the solutions to be regularly (or slowly) varying, are in fact necessary and sufficient for the nonoscillation of pertinent solutions. Such results are quite scarce in the existing literature in spite of the abundance of results on oscillation theory.

Part One

Linear Equations

Chapter 1

Existence of regular solutions

1.1. Preliminaries.

We consider second order linear equation

$$y'' + f(x)y = 0, \tag{1.1}$$

but the results are then easily generalized to more general case

$$y'' + g(x)y' + h(x)y = 0. \tag{1.2}$$

For the coefficient f in (1.1) it is assumed to be continuous on a half-axis $[a_0, \infty)$ for some $a_0 > 0$, and for $h(t)$, $g(t)$ in (1.2) it is required to be continuous and continously differentiable respectively.

In general f is of arbitrary sign. However some results deal with the special case $f(x) < 0$ when all (positive) solutions are convex. This opens possibilities for some additional results compared to the general case. Also, more direct methods can be used for the proofs.

Also, all solutions y are studied for $x > x_0 \geq a_0$.

All results of Part One which follow are essentialy related to one solution only, since the second linearly independent one, is then treated by the usual Wronskian technique i.e. by using well known formulae

$$y_2(x) = y_1(x)\int_a^x y_1^{-2}(t)dt, \quad \text{or} \quad y_2(x) = y_1(x)\int_x^\infty y_1^{-2}(t)dt \tag{1.3}$$

depending on the convergence of the integral. For the lower bound in all forthcoming integrals may be taken any real number a such that $a \geq x_0$. Of course, y_1 is of constant sign on the considered intervals.

In all proofs of results in secion 1.2 decreasing solutions play a dominant role. This is why we include here the following

Lemma 1.1. *Let for some* $a > 0$, $p \in C^1[a,\infty)$, $q \in C[a,\infty)$, $p(x) > 0$, $q(x) \geq 0$, *with* $q(x)$ *ultimately non-vanishing. Then the equation*

$$(p(x)y'(x))' = q(x)y(x)$$

has a positive decreasing solution on (x_0,∞) *for some* $x_0 > a$.

Among the various approaches regarding the proof we present the following one of M. Marini and P. Zezza [50].

Proof. First notice that for every nontrivial solution v of the considered equation there exists a $x_0 > a$ such that $v(t)$ is monotone on (x_0,∞). This follows since for the function $M(x) = p(x)v(x)v'(x)$, due to the positivity of $p(x)$ and $q(x)$ one has for $x \geq x_0$, $M'(x) = p(x)v'^2(x) + q(x)v^2(x) \geq 0$. Which implies that $v'(x)$ can have at most one zero greater then x_0.

Further divide the set of all relevant solutions into two classes

$$A = \{y = y(x) \text{ a solution: } \exists x_0 : y(x_0)y'(x_0) \geq 0\}$$

$$B = \{y = y(x) \text{ a solution: } \forall x \geq x_0,\ y(x)y'(x) < 0\}.$$

Without loss of generality we may consider solutions of class A as positive nondecreasing or negative nonincreasing and similarly for class B.

Class A is nonempty since it contains the solutions with positive initial conditions. Moreover one shows by the usual Wronskian technique that if a nonzero solution of class A is bounded, then all solutions of class A are such, [50, Lemma 2].

To complete the proof one has to show that there is a solution of class B.

Let v be any solution of the considered equation with positive initial conditions therefore belonging to class A. Then the function

$$u(x) = v(x)\int_{x_0}^{x} \frac{ds}{p(s)v^2(s)}$$

is another linearly independent solution.

Suppose first that $v(x)$, is unbounded (in fact - tending to infinity being increasing), and consider

$$\lim_{x\to\infty} \frac{u(x)}{v(x)} = \lim_{x\to\infty} \int_{x_0}^{x} \frac{ds}{p(s)v^2(s)} = K.$$