A.P.S. Selvadurai

Partial Differential Equations in Mechanics 2

力学中的偏微分方程 第2卷

The Biharmonic Equation
Poisson's Equation

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With 215 Figures

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To the memory of my parents

K.S. Selvadurai and W.M.A. Selvadurai

and to

Sally, Emily, Paul, Mark

and

Elizabeth

Preface

> *"For he who knows not mathematics cannot know any other sciences; what is more, he cannot discover his own ignorance or find its proper remedies."* [Opus Majus] Roger Bacon (1214-1294)

The material presented in these monographs is the outcome of the author's long-standing interest in the analytical modelling of problems in mechanics by appeal to the theory of partial differential equations. The impetus for writing these volumes was the opportunity to teach the subject matter to both undergraduate and graduate students in engineering at several universities. The approach is distinctly different to that which would adopted should such a course be given to students in pure mathematics; in this sense, the teaching of partial differential equations within an engineering curriculum should be viewed in the broader perspective of *"The Modelling of Problems in Engineering"*. An engineering student should be given the opportunity to appreciate how the various combination of balance laws, conservation equations, kinematic constraints, constitutive responses, thermodynamic restrictions, etc., culminates in the development of a partial differential equation, or sets of partial differential equations, with potential for applications to engineering problems. This ability to distill all the diverse information about a physical or mechanical process into partial differential equations is a particular attraction of the subject area. A second aspect of the teaching of partial differential equations to engineering students must cover topics that should enable them to pose an engineering problem as a correct mathematical statement. This process can include the mathematical structuring of the physical boundary conditions, regularity conditions, initial conditions and uniqueness theorems to generate a well-posed problem in partial differential equations. Thirdly, the presentation should include an introduction to the solution schemes that will highlight the basic structure of the solutions associated with the different classes of partial differential equations, by appeal to suitable idealizations of engineering problems. These volumes are also intended to illustrate the extensive range of applicability of the basic linear partial differential equations in a *multidisciplinary sense*, with special emphasis on applications to problems in mechanics encountered in civil engineering, mechanical engineering, theoretical and applied mechanics, chemical engineering, geological engineering, earth sciences, etc., covering topics such as fluid flow, diffusion and mass transport in porous media, pressure transients and moisture diffusion in porous geomaterials, heat conduction in solids, waves in elastic solids, fluids and membranes, elasto-mechanics of solids and structural elements and mechanics of viscous fluids.

These companion volumes contain a total of nine chapters, which introduce the basic concepts of partial differential equations with the central theme of modelling and applications in mechanics. The division of the presentation into two companion volumes with chapter continuity achieves two purposes. The material contained in Volume I, chapters 1 to 7, can form the subject matter of a senior level undergraduate course or a graduate level course in applications of partial differential equations in mechanics for engineering students. The material contained in Volume II, chapters 8 and 9, is ideally suited for a graduate level course devoted to applications of partial differential equations in mechanics of solids. The introductory chapter gives a review of the mathematical preliminaries, including vector calculus, Fourier series and integral transforms. This chapter is not intended to provide an exhaustive coverage of these topics and the interested reader can further review this material by consulting the bibliography cited at the end of these volumes. The presentation in chapter 1 is kept to a reasonable length by introducing as brief a derivation of the salient results and procedures as possible. Chapter 2 introduces partial differential equations and definitions of order, linearity, homogeneity, and well posedness. Chapter 3 provides a brief account of first-order partial differential equations with applications that involve characteristic equations. Applications of integral transform techniques to the solution of elementary problems involving transport in porous media are also discussed. Chapter 4 deals with the classification of partial differential equations of the second-order with procedures for their reduction to canonical forms. The application of Laplace's equation to problems of steady state heat conduction, ideal fluid flow, flow in porous media and applications to the study of deflections of stretched membranes are detailed in chapter 5. Chapter 6 examines the diffusion equation in relation to transient heat conduction, mass transport in porous media and pressure transients in porous media. Chapter 7 deals with the wave equation and its application to wave propagation in infinite and finite strings, vibrations of membranes and elementary one-dimensional vibration problems in solid mechanics. This chapter also examines the application of the wave equation to the study of shallow water waves. Chapter 8 presents a very complete exposition of the application of the biharmonic equation to problems in mechanics. A perusal of many existing texts on partial differential equations reveals the conspicuous absence of any detailed treatment of the biharmonic equation. Yet, the biharmonic equation is one of the most important partial differential equations in applied mechanics, with applications in the theory of elasticity, mechanics of elastic plates and the theory of slow flows of viscous fluids, all subject areas of fundamental importance to the engineering sciences. Chapter 9 deals with Poisson's equation. Many expositions of Poisson's equation present themselves as appendages to Laplace's equation. The objective of this chapter is to demonstrate that Poisson's equation has significance in its own right and has extensive applications to engineering problems dealing with steady state heat conduction in heat generating media, groundwater flow with recharge or depletion, mechanics of stretched, loaded membranes and in the study of the theory of torsion of prismatic elastic bodies.

Each chapter contains a detailed discussion of the application of the relevant partial differential equation, its derivation in a generalized fashion and the formulation of consistent boundary and/or initial conditions required for their well posedness. The proof of the relevant uniqueness theorems, maximum principles and other topics of general interest to identifying the qualitative aspects of the specific partial differential equations are also discussed. Worked examples within the text and problem sets at the end of each chapter highlight engineering applications of the theories and relevant analytical developments. The volumes are reasonably self-contained, in the sense that all necessary material, including developments of the governing partial differential equations, proofs of general theorems and applications, are presented in their entirety without recourse to references within the text. There is, of course, a wealth of information available in other texts and treatises devoted to the subject of partial differential equations and the reader, and students in particular, should at least aware of these developments. To this end, these volumes contain an extensive bibliography divided into topics covering general engineering mathematics, ordinary differential equations, partial differential equations, Fourier series, integral transforms and special functions, boundary value problems and mathematical methods. The bibliography also contains titles of interest to the separate chapters, including first-order partial differential equations, Laplace's equation, the diffusion equation, the wave equation, the biharmonic equation and Poisson's equation. Although not required for the subject matter covered in these volumes, the bibliography also contains titles related to non-linear partial differential equations, numerical methods for the solution of partial differential equations, applications of computer based symbolic manipulation methods to mathematics and references to historical material in both mechanics and mathematics. Symbolic computer methods are gaining popularity in many engineering curricula; they should be considered as a useful complement to carrying out mathematical operations in a time effective manner, particularly as computer laboratory exercises of examples given in these volumes.

These monographs, by design, emphasize analytical procedures for the solution of the various partial differential equations. This should not be construed as an opportunity to de-emphasize numerical and computational techniques. On the contrary, any realistic engineering application of even the simplest of theories, such as those described by the linear partial differential equations treated in these monographs will invariably require access to sophisticated numerical schemes. These schemes can include finite difference, finite element and boundary integral equation techniques that are gaining considerable popularity in engineering curricula, both at the undergraduate and graduate levels. The teaching of these numerical techniques can only benefit by instilling in undergraduate engineering students the confidence in both the mathematical aspects of partial differential equations and their applications potential. Indeed, in advanced finite element formulations based on the Galerkin technique, the governing partial differential equations are a prerequisite; similarly, knowledge of the Green's function applicable to a particular partial differential equation is a requirement for the application of boundary integral equation techniques. The subject of partial differential equations has a long and rich tradition in mathematics and mechanics, and, as the bibliography demonstrates, has at its disposal an extensive collection of texts and treatises devoted to the subject. The bibliography is certainly not meant to be all encompassing and up to date. The texts cited are considered sufficient as supplementary reading material. The present volumes, however, differ from many traditional presentations in that the subject matter is introduced within the context of mathematical rigour, modelling in mechanics and the applications of elementary solutions to problems in engineering mechanics.

The need for engineers to understand fundamental concepts associated with partial differential equations and techniques available for their solution becomes apparent when mathematical modelling and analysis of engineering problems are set in their modern context. Engineers are constantly exposed to new engineering software tools, usually involving numerical methods that enable them to carry out engineering computations for practical problems with speed and accuracy. In a majority of these cases, the numerical schemes are designed to solve partial differential equations with complicated coupled phenomena, which usually have a non-linear character. The validity and success of the numerical procedures in such software tools cannot be confirmed in a universal sense. The opportunity, however, exists for engineers to conduct calibrations of such numerical schemes by recourse to solutions developed for classes of linearized problems, albeit simplified. The modelling of an engineering problem as a mathematical statement can be regarded as the modern equivalent of a design exercise. These companion volumes highlight the point that even elementary partial differential equations are a powerful tool for posing and solving practical problems in engineering. The mathematical rigour of the presentations is balanced by an extensive variety of applications in mechanics. The practical nature of the problems examined emphasizes mathematical modelling as an important aspect of the training of engineering students. In this sense, the subject of partial differential equations has much to offer to enhance the quality of the broader mathematical education of students in engineering and to effectively integrate mathematics into the mainstream of modelling of engineering problems.

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I would like to thank Dr. Dietrich Merkle, Engineering Editor, at Springer-Verlag, Heidelberg and his staff for their very positive support for the project and for the assistance they have provided in allowing me to present the manuscript in a form that would be of benefit to the readership. Although I had the pleasure of "writing" the manuscript the arduous task of preparing several versions of it into LaTeX forms was performed by Ms. Livia Nardini. Mr. Nick Vannelli converted my "sketches" to Corel Draw equivalents. I am also grateful to a number of technical assistants, notably, Marwan Al-Habbal and Todd Hirtle who were involved in preparing the final manuscripts to conform to Springer guidelines. As former students in the PDEs course, their familiarity with the course material was a great asset in this exercise. Every attempt has been made to check the accuracy of the material contained in these volumes. Some errors are bound to exist despite my best efforts. I would therefore be most grateful to the readers if they would be kind enough to bring these to my attention.

Last but not least, my appreciation goes to my wife Sally, not only for her editorial assistance, but also for her willingness to attend to the demands of a family when I was engrossed, absent or preoccupied with the task of completing these volumes against self-imposed deadlines. It is fair to say that all the family appreciates the completion of the project.

Montréal, April 2000 *A.P.S. Selvadurai*

Contents

Chapter 8

The biharmonic equation

An important common theme in the developments presented in connection with Laplace's equation, the diffusion equation and the wave equation is that they are all of the second-order and represent the fundamental equations which govern elliptic, parabolic and hyperbolic partial differential equations, respectively. A further general observation in previous expositions is that as the phenomena that are being modelled becomes either more complex or encompasses more complicated fundamental processes, the partial differential equations which describe such phenomena are expected to acquire a higher order. This was evident in the description of advection-diffusion phenomena governing the transport of chemicals in porous media. In the presence of only advective phenomena the transport process can be described by a first-order partial differential equation; when diffusive processes are taken into consideration, the transport process can be described by a second-order partial differential equation. The biharmonic equation is one such partial differential equation which arises as a result of modelling more complex phenomena encountered in problems in science and engineering. The term biharmonic is indicative of the fact that the function describing the processes satisfies Laplace's equation twice explicitly. The exact first usage of the biharmonic equation is not entirely clear since every harmonic function which satisfies Laplace's equation is also a biharmonic function. Many of the applications of the biharmonic equation stem from the consideration of more complex mechanical and physical processes involving solids and fluids. One of the earliest applications of the biharmonic equation deals with the classical theory of flexure of elastic plates developed, among others, by J. Bernoulli (1667-1748), Euler (1707-1831), Lagrange (1736-1813), Germain (1776-1831), Poisson (1781-1840), Navier (1785-1836), Cauchy (1789-1857) and Lamé (1795-1870). Developments to the mathematical modelling of the theory of plates continued with contributions by Kirchhoff (1824-1887), Levy