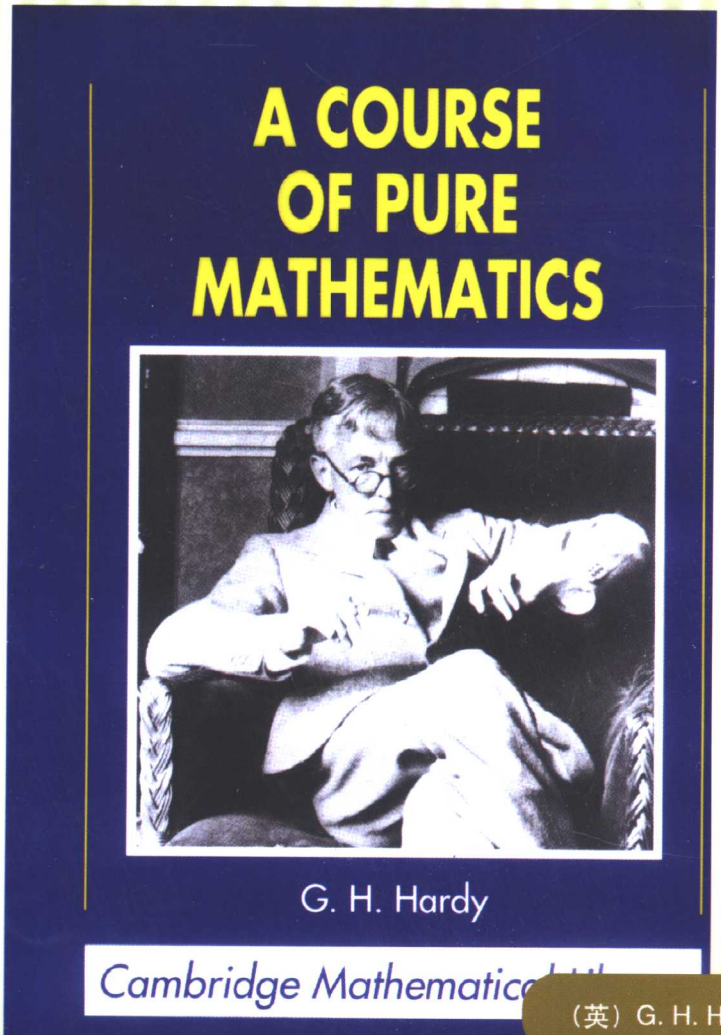
HZ BOOKS

经 典 原 版 书 库

纯数学教程

（英文版 · 第10版）



（英）G. H. Hardy 著

机械工业出版社
China Machine Press

经典原版书库

纯数学教程

（英文版·第10版）

A Course of Pure Mathematics

(Tenth Edition)

（英）G. H. Hardy 著

机械工业出版社
China Machine Press

G. H. Hardy: A Course of Pure Mathematics, Tenth Edition (ISBN: 0-521-09227-2).

Originally published by Cambridge University Press in 2002.

This reprint edition is published with the permission of the Syndicate of the Press of the University of Cambridge, Cambridge, England.

Copyright © 2002 by Cambridge University Press.

This edition is licensed for distribution and sale in the People's Republic of China only, excluding Hong Kong, Taiwan and Macao and may not be distributed and sold elsewhere.

本书原版由剑桥大学出版社出版。

本书英文影印版由英国剑桥大学出版社授权出版。

此版本仅限在中华人民共和国境内（不包括中国香港、台湾、澳门地区）销售发行，未经授权的本书出口将被视为违反版权法的行为。

版权所有，侵权必究。

本书版权登记号：图字：01-2004-0296

图书在版编目（CIP）数据

纯数学教程（英文版·第10版）/（英）哈代（Hardy, G. H.）著. -北京：机械工业出版社，2004.2

（经典原版书库）

书名原文：A Course of Pure Mathematics , Tenth Edition

ISBN 7-111-13785-X

Ⅰ. 纯… Ⅱ. 哈… Ⅲ. 高等数学-教材-英文 Ⅳ. O13

中国版本图书馆CIP数据核字（2003）第126640号

机械工业出版社（北京市西城区百万庄大街22号 邮政编码 100037）

责任编辑：迟振春

北京瑞德印刷有限公司印刷·新华书店北京发行所发行

2004 年 2 月第 1 版第 1 次印刷

787mm × 1092mm 1/16 · 32.5印张

印数：0 001-2 000册

定价：65.00元

凡购本书，如有倒页、脱页、缺页，由本社发行部调换

本社购书热线：(010) 68326294

PREFACE TO THE TENTH EDITION

THE changes in the present edition are as follows:

1. An index has been added. Hardy had begun a revision of an index compiled by Professor S. Mitchell; this has been completed, as far as possible on Hardy's lines, by Dr T. M. Flett.

2. The original proof of the Heine-Borel Theorem (pp. 197–199) has been replaced by two alternative proofs due to Professor A. S. Besicovitch.

3. Example 24, p. 394 has been added to.

August, 1950 J. E. LITTLEWOOD

PREFACE TO THE SEVENTH EDITION

THE changes in this edition are more important than in any since the second. The book has been reset, and this has given me the opportunity of altering it freely.

I have cancelled what was Appendix II (on the 'O, o, $\sim$' notation), and incorporated its contents in the appropriate places in the text. I have rewritten the parts of Chs. VI and VII which deal with the elementary properties of differential coefficients. Here I have found de la Vallée-Poussin's *Cours d'analyse* the best guide, and I am sure that this part of the book is much improved. These important changes have naturally involved many minor emendations.

I have inserted a large number of new examples from the papers for the Mathematical Tripos during the last twenty years, which should be useful to Cambridge students. These were collected for me by Mr E. R. Love, who has also read all the proofs and corrected many errors.

The general plan of the book is unchanged. I have often felt tempted, re-reading it in detail for the first time for twenty years, to make much more drastic changes both in substance and in style. It was written when analysis was neglected in Cambridge, and with an emphasis and enthusiasm which seem rather ridiculous now. If I were to rewrite it now I should not write (to use Prof. Littlewood's simile) like 'a missionary talking to cannibals', but with decent terseness and restraint; and, writing more shortly, I should be able to include a great deal more. The book would then be much more like a *Traité d'analyse* of the standard pattern.

It is perhaps fortunate that I have no time for such an undertaking, since I should probably end by writing a much better but much less individual book, and one less useful as an introduction to the books on analysis of which, even in England, there is now no lack.

November, 1937 G. H. H.

EXTRACT FROM THE PREFACE TO THE FIRST EDITION

THIS book has been designed primarily for the use of first year students at the Universities whose abilities reach or approach something like what is usually described as 'scholarship standard'. I hope that it may be useful to other classes of readers, but it is this class whose wants I have considered first. It is in any case a book for mathematicians: I have nowhere made any attempt to meet the needs of students of engineering or indeed any class of students whose interests are not primarily mathematical.

I regard the book as being really elementary. There are plenty of hard examples (mainly at the ends of the chapters): to these I have added, wherever space permitted, an outline of the solution. But I have done my best to avoid the inclusion of anything that involves really difficult ideas.

September, 1908 G. H. H.

试读结束：需要全本请在线购买： www.ertongbook.com

华章教育 赤诚奉献

经典原版书库（相关部分）

具体数学：计算机科学基础（第2版）	Graham/49.00
组合数学（第3版）	Brualdi/35.00
离散数学及其应用（第5版）	Rosen/79.00
离散数学及其应用（第4版）	Rosen/59.00
高等微积分	Fitzpatrick/69.00
金融数学	Stampfli/35.00
数值分析（第3版）	Kincaid/75.00
概率论及其在投资、保险、工程中的应用	Bean/55.00
代数	Isaacs/65.00
傅里叶分析与小波分析导论	Pinsky/49.00
数学建模（第3版）	Giordano/59.00（附光盘）
微分方程与边界值问题（第5版）	Zill/69.00
应用回归分析和其他多元方法	Kleinbaum/88.00
概率统计	Stone/89.00
多元数据分析	Lattin/69.00（附光盘）
数理统计和数据分析	Rice/78.00
随机过程导论	Kao/49.00
预测与时间序列（第3版）	Bowerman/89.00
线性代数	Jain/49.00（附光盘）
复变函数及应用（第7版）	Brown/42.00
实分析与复分析（第3版）	Rudin/39.00
数学分析原理（第3版）	Rudin/35.00
泛函分析（第2版）	Rudin/42.00
复分析（第3版）	Ahlfors/35.00
逼近论教程	Cheney/39.00

计算机科学丛书（相关部分）

离散数学及其应用（原书第4版）	Rosen/袁崇义/75.00
组合数学（原书第3版）	Brualdi/冯舜玺/38.00

重点大学计算机教材（相关部分）

数值方法	金一庆/25.00

CONTENTS

(Entries in small print at the end of the contents of each chapter refer to subjects discussed incidentally in the examples)

CHAPTER I

REAL VARIABLES

SECT.		PAGE
1–2.	Rational numbers	1
3–7.	Irrational numbers	3
8.	Real numbers	14
9.	Relations of magnitude between real numbers	16
10–11.	Algebraical operations with real numbers	17
12.	The number $\sqrt{2}$	20
13–14.	Quadratic surds	20
15.	The continuum	24
16.	The continuous real variable	27
17.	Sections of the real numbers. Dedekind's theorem	28
18.	Points of accumulation	30
19.	Weierstrass's theorem	31
	Miscellaneous examples	32

Decimals, 1. Gauss's theorem, 7. Graphical solution of quadratic equations, 21. Important inequalities, 33. Arithmetical and geometrical means, 34. Cauchy's inequality, 34. Cubic and other surds, 36. Algebraical numbers, 38.

CHAPTER II

FUNCTIONS OF REAL VARIABLES

20.	The idea of a function	40
21.	The graphical representation of functions. Coordinates	43
22.	Polar coordinates	45
23.	Polynomials	46
24–25.	Rational functions	49
26–27.	Algebraical functions	52
28–29.	Transcendental functions	55
30.	Graphical solution of equations	60
31.	Functions of two variables and their graphical representation	61
32.	Curves in a plane	62
33.	Loci in space	63
	Miscellaneous examples	67

Trigonometrical functions, 55. Arithmetical functions, 58. Cylinders, 64. Contour maps, 64. Cones, 65. Surfaces of revolution, 65. Ruled surfaces, 66. Geometrical constructions for irrational numbers, 68. Quadrature of the circle, 70.

CHAPTER III

COMPLEX NUMBERS

SECT.		PAGE
34–38.	Displacements	72
39–42.	Complex numbers	80
43.	The quadratic equation with real coefficients	84
44.	Argand's diagram	87
45.	De Moivre's theorem	88
46.	Rational functions of a complex variable	90
47–49.	Roots of complex numbers	101
	Miscellaneous examples	104

Properties of a triangle, 92, 104. Equations with complex coefficients, 94. Coaxal circles, 96. Bilinear and other transformations, 97, 100, 107. Cross ratios, 99. Condition that four points should be concyclic, 100. Complex functions of a real variable, 100. Construction of regular polygons by Euclidean methods, 103. Imaginary points and lines, 106.

CHAPTER IV

LIMITS OF FUNCTIONS OF A POSITIVE INTEGRAL VARIABLE

50.	Functions of a positive integral variable	110
51.	Interpolation	111
52.	Finite and infinite classes	112
53–57.	Properties possessed by a function of n for large values of n	113
58–61.	Definition of a limit and other definitions	120
62.	Oscillating functions	126
63–68.	General theorems concerning limits	129
69–70.	Steadily increasing or decreasing functions	136
71.	Alternative proof of Weierstrass's theorem	138
72.	The limit of x^n	139
73.	The limit of $\left(1+\frac{1}{n}\right)^n$	142
74.	Some algebraical lemmas	143
75.	The limit of $n(\sqrt[n]{x}-1)$	144
76–77.	Infinite series	145
78.	The infinite geometrical series	149
79.	The representation of functions of a continuous real variable by means of limits	153
80.	The bounds of a bounded aggregate	155
81.	The bounds of a bounded function	156
82.	The limits of indetermination of a bounded function	156
83–84.	The general principle of convergence	158
85–86.	Limits of complex functions and series of complex terms	160

SECT.		PAGE
87–88.	Applications to z^n and the geometrical series	162
89.	The symbols O, o, $\sim$	164
	Miscellaneous examples	166

Oscillation of $\sin n\theta\pi$, 125, 127, 158. Limits of $n^k x^n$, $\sqrt[n]{x}$, $\sqrt[n]{n}$, $\frac{x^n}{n!}$, $\sqrt[n]{(n!)}$, $\binom{m}{n} x^n$, 141, 144. Decimals, 149. Arithmetic series, 152. Harmonic series, 153. Equation $x_{n+1}=f(x_n)$, 166. Limit of a mean value, 167. Expansions of rational functions, 170.

CHAPTER V

LIMITS OF FUNCTIONS OF A CONTINUOUS VARIABLE. CONTINUOUS AND DISCONTINUOUS FUNCTIONS

90–92.	Limits as $x \to \infty$ or $x \to -\infty$	172
93–97.	Limits as $x \to a$	175
98.	The symbols O, o, $\sim$: orders of smallness and greatness	183
99–100.	Continuous functions of a real variable	185
101–105.	Properties of continuous functions. Bounded functions. The oscillation of a function in an interval	190
106–107.	Sets of intervals on a line. The Heine-Borel theorem	196
108.	Continuous functions of several variables	201
109–110.	Implicit and inverse functions	203
	Miscellaneous examples	206

Limits and continuity of polynomials and rational functions, 179, 187. Limit of $\frac{x^m - a^m}{x-a}$, 181. Limit of $\frac{\sin x}{x}$, 182. Infinity of a function, 188. Continuity of $\cos x$ and $\sin x$, 188. Classification of discontinuities, 188. Semi-continuity, 209.

CHAPTER VI

DERIVATIVES AND INTEGRALS

111–113.	Derivatives	210
114.	General rules for differentiation	216
115.	Derivatives of complex functions	218
116.	The notation of the differential calculus	218
117.	Differentiation of polynomials	220
118.	Differentiation of rational functions	223
119.	Differentiation of algebraical functions	224
120.	Differentiation of transcendental functions	225
121.	Repeated differentiation	228
122.	General theorems concerning derivatives. Rolle's theorem	231
123–125.	Maxima and minima	234
126–127.	The mean value theorem	242
128.	Cauchy's mean value theorem	244

SECT.		PAGE
129.	A theorem of Darboux	245
130–131.	Integration. The logarithmic function	245
132.	Integration of polynomials	249
133–134.	Integration of rational functions	250
135–142.	Integration of algebraical functions. Integration by rationalisation. Integration by parts	254
143–147.	Integration of transcendental functions	264
148.	Areas of plane curves	268
149.	Lengths of plane curves	270
	Miscellaneous examples	273

Derivative of x^m, 214. Derivatives of $\cos x$ and $\sin x$, 214. Tangent and normal to a curve, 214, 228. Multiple roots of equations, 221, 277. Rolle's theorem for polynomials, 222. Leibniz's theorem, 229. Maxima and minima of the quotient of two quadratics, 238, 277. Axes of a conic, 241. Lengths and areas in polar coordinates, 273. Differentiation of a determinant, 274. Formulae of reduction, 282.

CHAPTER VII

ADDITIONAL THEOREMS IN THE DIFFERENTIAL AND INTEGRAL CALCULUS

150–151.	Taylor's theorem	285
152.	Taylor's series	291
153.	Applications of Taylor's theorem to maxima and minima	293
154.	The calculation of certain limits	293
155.	The contact of plane curves	296
156–158.	Differentiation of functions of several variables	300
159.	The mean value theorem for functions of two variables	305
160.	Differentials	307
161–162.	Definite integrals	311
163.	The circular functions	316
164.	Calculation of the definite integral as the limit of a sum	319
165.	General properties of the definite integral	320
166.	Integration by parts and by substitution	324
167.	Alternative proof of Taylor's theorem	327
168.	Application to the binomial series	328
169.	Approximate formulae for definite integrals. Simpson's rule	328
170.	Integrals of complex functions	331
	Miscellaneous examples	332

Newton's method of approximation to the roots of equations, 288. Series for $\cos x$ and $\sin x$, 292. Binomial series, 292. Tangent to a curve, 298, 310, 335. Points of inflexion, 298. Curvature, 299, 334. Osculating conics, 299, 334. Differentiation of implicit functions, 310. Maxima and minima of functions of two variables, 311. Fourier's integrals, 318, 323. The second mean value theorem, 325. Homogeneous functions, 334. Euler's theorem, 334. Jacobians, 335. Schwarz's inequality, 340.

CHAPTER VIII

THE CONVERGENCE OF INFINITE SERIES AND INFINITE INTEGRALS

SECT.		PAGE
171–174.	Series of positive terms. Cauchy's and d'Alembert's tests of convergence	341
175.	Ratio tests	343
176.	Dirichlet's theorem	347
177.	Multiplication of series of positive terms	347
178–180.	Further tests for convergence. Abel's theorem. Maclaurin's integral test	349
181.	The series Σn^{-s}	352
182.	Cauchy's condensation test	354
183.	Further ratio tests	355
184–189.	Infinite integrals	356
190.	Series of positive and negative terms	371
191–192.	Absolutely convergent series	373
193–194.	Conditionally convergent series	375
195.	Alternating series	376
196.	Abel's and Dirichlet's tests of convergence	379
197.	Series of complex terms	381
198–201.	Power series	382
202.	Multiplication of series	386
203.	Absolutely and conditionally convergent infinite integrals	388
	Miscellaneous examples	390

The series $\Sigma n^k r^n$ and allied series, 345. Hypergeometric series, 355. Binomial series, 356, 386, 387. Transformation of infinite integrals by substitution and integration by parts, 361, 363, 369. The series $\Sigma a_n \cos n\vartheta$, $\Sigma a_n \sin n\vartheta$, 374, 380, 381. Alteration of the sum of a series by rearrangement, 378. Logarithmic series, 385. Multiplication of conditionally convergent series, 388, 394. Recurring series, 392. Difference equations, 393. Definite integrals, 395.

CHAPTER IX

THE LOGARITHMIC, EXPONENTIAL, AND CIRCULAR FUNCTIONS OF A REAL VARIABLE

204–205.	The logarithmic function	398
206.	The functional equation satisfied by $\log x$	401
207–209.	The behaviour of $\log x$ as x tends to infinity or to zero	402
210.	The logarithmic scale of infinity	403
211.	The number e	405
212–213.	The exponential function	406
214.	The general power a^x	409
215.	The exponential limit	410
216.	The logarithmic limit	411

试读结束：需要全本请在线购买： www.ertongbook.com

SECT.		PAGE
217.	Common logarithms	412
218.	Logarithmic tests of convergence	417
219.	The exponential series	422
220.	The logarithmic series	425
221.	The series for arc tan x	426
222.	The binomial series	429
223.	Alternative development of the theory	431
224–226.	The analytical theory of the circular functions	432
	Miscellaneous examples	438

Integrals containing the exponential function, 413. The hyperbolic functions, 415. Integrals of certain algebraical functions, 416. Euler's constant, 420. Irrationality of e, 423. Approximation to surds by the binomial theorem, 430. Irrationality of $\log_{10} n$, 438. Definite integrals, 445, 446.

CHAPTER X

THE GENERAL THEORY OF THE LOGARITHMIC, EXPONENTIAL, AND CIRCULAR FUNCTIONS

227–228.	Functions of a complex variable	447
229.	Curvilinear integrals	448
230.	Definition of the logarithmic function	449
231.	The values of the logarithmic function	451
232–234.	The exponential function	456
235–236.	The general power a^z	457
237–240.	The trigonometrical and hyperbolic functions	462
241.	The connection between the logarithmic and inverse trigonometrical functions	466
242.	The exponential series	468
243.	The series for cos z and sin z	469
244–245.	The logarithmic series	471
246.	The exponential limit	474
247.	The binomial series	476
	Miscellaneous examples	479

The functional equation satisfied by Log z, 454. The function e^z, 460. Logarithms to any base, 461. The inverse cosine, sine, and tangent of a complex number, 464. Trigonometrical series, 470, 472–474, 484, 485. Roots of transcendental equations, 479, 480. Transformations, 480–483. Stereographic projection, 482. Mercator's projection, 482. Level curves, 484–485. Definite integrals, 486.

APPENDIX I.	The proof that every equation has a root	487
APPENDIX II.	A note on double limit problems	493
APPENDIX III.	The infinite in analysis and geometry	497
APPENDIX IV.	The infinite in analysis and geometry	502
INDEX		505

CHAPTER I

REAL VARIABLES

1. Rational numbers. A fraction $r = p/q$, where p and q are positive or negative integers, is called a *rational number*. We can suppose (i) that p and q have no common factor, since if they have a common factor we can divide each of them by it, and (ii) that q is positive, since

$$p/(-q) = (-p)/q, \quad (-p)/(-q) = p/q.$$

To the rational numbers thus defined we may add the 'rational number 0' obtained by taking $p = 0$.

We assume that the reader is familiar with the ordinary arithmetical rules for the manipulation of rational numbers. The examples which follow demand no knowledge beyond this.

Examples I. 1. If r and s are rational numbers, then $r+s$, $r-s$, rs, and r/s are rational numbers, unless in the last case $s = 0$ (when r/s is of course meaningless).

2. If λ, m, and n are positive rational numbers, and $m > n$, then $\lambda(m^2-n^2)$, $2\lambda mn$, and $\lambda(m^2+n^2)$ are positive rational numbers. Hence show how to determine any number of right-angled triangles the lengths of all of whose sides are rational.

3. Any terminated decimal represents a rational number whose denominator contains no factors other than 2 or 5. Conversely, any such rational number can be expressed, and in one way only, as a terminated decimal.

[The general theory of decimals will be considered in Ch. IV.]

4. The positive rational numbers may be arranged in the form of a simple series as follows:

$$\tfrac{1}{1}, \tfrac{2}{1}, \tfrac{1}{2}, \tfrac{3}{1}, \tfrac{2}{2}, \tfrac{1}{3}, \tfrac{4}{1}, \tfrac{3}{2}, \tfrac{2}{3}, \tfrac{1}{4}, \ldots.$$

Show that p/q is the $[\frac{1}{2}(p+q-1)(p+q-2)+q]$th term of the series.

[In this series every rational number is repeated indefinitely. Thus 1 occurs as $\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \ldots$. We can of course avoid this by omitting every number

which has already occurred in a simpler form, but then the problem of determining the precise position of p/q becomes more complicated.]

2. The representation of rational numbers by points on a line. It is convenient, in many branches of mathematical analysis, to make a good deal of use of geometrical illustrations.

The use of geometrical illustrations in this way does not, of course, imply that analysis has any sort of dependence upon geometry: they are illustrations and nothing more, and are employed merely for the sake of clearness of exposition. This being so, it is not necessary that we should attempt any logical analysis of the ordinary notions of elementary geometry; we may be content to suppose, however far it may be from the truth, that we know what they mean.

Assuming, then, that we know what is meant by a *straight line*, a *segment* of a line, and the *length* of a segment, let us take a straight line A, produced indefinitely in both directions, and a segment A_0A_1 of any length. We call A_0 the *origin*, or *the point* 0, and A_1 *the point* 1, and we regard these points as representing the numbers 0 and 1.

In order to obtain a point which shall represent a positive rational number $r = p/q$, we choose the point A_r such that

$$A_0A_r/A_0A_1 = r,$$

A_0A_r being a stretch of the line extending in the same direction along the line as A_0A_1, a direction which we shall suppose to be from left to right when, as in Fig. 1, the line is drawn horizontally across the paper. In order to obtain a point to represent a

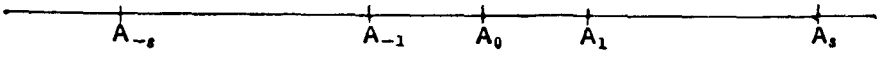


Fig. 1

negative rational number $r = -s$, it is natural to regard length as a magnitude capable of sign, positive if the length is measured in one direction (that of A_0A_1), and negative if measured in the other, so that $AB = -BA$; and to take as the point representing r the point A_{-s} such that

$$A_0A_{-s} = -A_{-s}A_0 = -A_0A_s.$$

We thus obtain a point A_r on the line corresponding to every rational value of r, positive or negative, and such that

$$A_0A_r = r.A_0A_1;$$

and if, as is natural, we take A_0A_1 as our unit of length, and write $A_0A_1 = 1$, then we have

$$A_0A_r = r.$$

We shall call the points A_r the *rational points* of the line.

3. Irrational numbers. If the reader will mark off on the line all the points corresponding to the rational numbers whose denominators are 1, 2, 3, ... in succession, he will readily convince himself that he can cover the line with rational points as closely as he likes. We can state this more precisely as follows: *if we take any segment BC on Λ, we can find as many rational points as we please on BC.*

Suppose, for example, that BC falls within the segment A_1A_2. It is evident that if we choose a positive integer k so that

$$k.BC > 1 \quad \text{.....(1)*},$$

and divide A_1A_2 into k equal parts, then at least one of the points of division (say P) must fall inside BC, without coinciding with either B or C. For if this were not so, BC would be entirely included in one of the k parts into which A_1A_2 has been divided, which contradicts the supposition (1). But P obviously corresponds to a rational number whose denominator is k. Thus at least one rational point P lies between B and C. But then we can find another such point Q between B and P, another between B and Q, and so on indefinitely; i.e., as we asserted above, we can find as many as we please. We may express this by saying that BC includes *infinitely many* rational points.

The meaning of such phrases as '*infinitely many*' or '*an infinity of*', in such sentences as 'BC includes infinitely many rational points' or 'there are an infinity of rational points on BC' or 'there are an infinity of positive integers', will be considered more closely in Ch. IV. The assertion 'there are an infinity of positive integers' means 'given any positive integer n,

* The assumption that this is possible is equivalent to the assumption of what is known as the axiom of Archimedes.

however large, we can find more than n positive integers'. This is plainly true whatever n may be, e.g. for $n = 100{,}000$ or $100{,}000{,}000$. The assertion means exactly the same as 'we can find *as many positive integers as we please*'.

The reader will easily convince himself of the truth of the following assertion, which is substantially equivalent to what was proved in the second paragraph of this section: given any rational number r, and any positive integer n, we can find another rational number lying on either side of r and differing from r by less than $1/n$. It is merely to express this differently to say that we can find a rational number lying on either side of r and differing from r by *as little as we please*. Again, given any two rational numbers r and s, we can interpolate between them a chain of rational numbers in which any two consecutive terms differ by as little as we please, that is to say by less than $1/n$, where n is any positive integer assigned beforehand.

From these considerations the reader might be tempted to infer that an adequate view of the nature of the line could be obtained by imagining it to be formed simply by the rational points which lie on it. And it is certainly the case that if we imagine the line to be made up solely of the rational points, and all other points (if there are any such) to be eliminated, the figure which remained would possess most of the properties which common sense attributes to the straight line, and would, to put the matter roughly, look and behave very much like a line.

A little further consideration, however, shows that this view would involve us in serious difficulties.

Let us look at the matter for a moment with the eye of common sense, and consider some of the properties which we may reasonably expect a straight line to possess if it is to satisfy the idea which we have formed of it in elementary geometry.

The straight line must be composed of points, and any segment of it by all the points which lie between its end points. With any such segment must be associated a certain entity called its *length*, which must be a *quantity* capable of *numerical measurement* in terms of any standard or unit length, and these lengths must be capable of combination with one another, according to the ordinary rules of algebra, by means of addition or multiplication.

Again, it must be possible to construct a line whose length is the sum or product of any two given lengths. If the length PQ, along a given line, is a, and the length QR, along the same straight line, is b, the length PR must be $a+b$. Moreover, if the lengths OP, OQ, along one straight line, are 1 and a, and the length OR along another straight line is b, and if we determine the length OS by Euclid's construction (Euc. VI. 12) for a fourth proportional to the lines OP, OQ, OR, this length must be ab, the algebraical fourth proportional to $1, a, b$. And it is hardly necessary to remark that the sums and products thus defined must obey the ordinary 'laws of algebra'; viz.

$$a+b=b+a, \quad a+(b+c)=(a+b)+c,$$
$$ab=ba, \quad a(bc)=(ab)c, \quad a(b+c)=ab+ac.$$

The lengths of our lines must also obey a number of obvious laws concerning inequalities as well as equalities: thus if A, B, C are three points lying along Λ from left to right, we must have $AB<AC$, and so on. Moreover it must be possible, on our fundamental line Λ, to find a point P such that A_0P is equal to any segment whatever taken along Λ or along any other straight line. All these properties of a line, and more, are involved in the presuppositions of our elementary geometry.

Now it is very easy to see that the idea of a straight line as composed of a series of points, each corresponding to a rational number, cannot possibly satisfy all these requirements. There

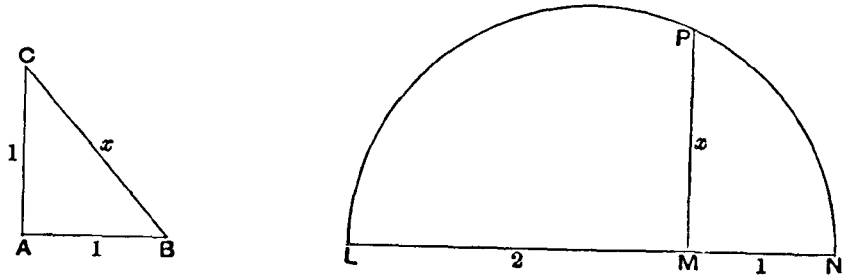


Fig. 2

are various elementary geometrical constructions, for example, which purport to construct a length x such that $x^2=2$. For instance, we may construct an isosceles right-angled triangle

试读结束：需要全本请在线购买： www.ertongbook.com