

D. J. H. GARLING

A COURSE IN  
**Mathematical  
Analysis**

VOLUME III  
Complex Analysis,  
Measure and Integration

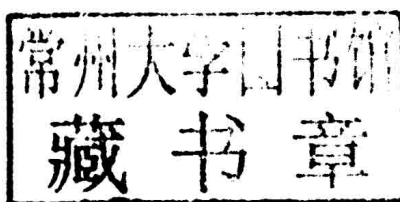
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# A COURSE IN MATHEMATICAL ANALYSIS

Volume III  
Complex Analysis,  
Measure and Integration

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*Emeritus Reader in Mathematical Analysis,  
University of Cambridge, and  
Fellow of St John's College, Cambridge*



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# A COURSE IN MATHEMATICAL ANALYSIS

## Volume III: Complex Analysis, Measure and Integration

The three volumes of *A Course in Mathematical Analysis* provide a full and detailed account of all those elements of real and complex analysis that an undergraduate mathematics student can expect to encounter in the first two or three years of study. Containing hundreds of exercises, examples and applications, these books will become an invaluable resource for both students and instructors.

Volume I focuses on the analysis of real-valued functions of a real variable. Volume II goes on to consider metric and topological spaces, and functions of a vector variable, and includes an introduction to the theory of manifolds in Euclidean space. This third volume develops the classical theory of functions of a complex variable. It carefully establishes the properties of the complex plane, including a proof of the Jordan curve theorem. Lebesgue measure is introduced, and is used as a model for other measure spaces, where the theory of integration is developed. The Radon–Nikodym theorem is proved, and the differentiation of measures is discussed.

D. J. H. GARLING is Emeritus Reader in Mathematical Analysis at the University of Cambridge and Fellow of St. John's College, Cambridge. He has fifty years' experience of teaching undergraduate students in most areas of pure mathematics, but particularly in analysis.



# Introduction

This book is the third and final volume of a full and detailed course in the elements of real and complex analysis that mathematical undergraduates may expect to meet. Indeed, I have based it on those parts of analysis that undergraduates at Cambridge University meet, or used to meet, in their first two years. I have however found it desirable to go rather further in certain places, in order to give a rounded account of the material.

In Part Five, we develop the theory of functions of a complex variable. To begin with, we consider holomorphic functions (functions which are complex-differentiable) and analytic functions (functions which can be defined by power series), and the results seem similar to those of real case. Things change when path-integrals are introduced. To use these, a good understanding of the topology of the plane is needed. We give a careful account of this, including a proof of the Jordan curve theorem (every simple closed curve has an inside and an outside). With this in place, various forms of Cauchy's theorem and Cauchy's integral formula are proved. These lead on to many magical results. Chapter 25 is geometric. A single-valued holomorphic function is conformal (that is, it preserves angles and orientations). We consider the problem of mapping one domain conformally onto another, and end by proving the celebrated Riemann mapping theorem, which says that if  $U$  and  $V$  are domains in the complex plane which are proper subsets of the plane and are simply-connected (there are no holes) then there exists a conformal mapping of  $U$  onto  $V$ . In Chapter 26, we apply the theory that we have developed to various problems, some of which were first introduced in Volume I.

In Volume I, we developed properties of the Riemann integral. This is very satisfactory when we wish to integrate continuous or monotonic functions, and is a useful precursor for the complex path integrals that we consider in Part Five, but it has serious shortcomings. In Part Six, we introduce

Lebesgue measure on the real line. Abstract measure theory is a large and important subject, but the topological properties of the real line make the construction of Lebesgue measure on the real line rather straightforward. With this example in place, we introduce the notion of a measure space, and the corresponding space of measurable functions. This then leads on easily to the theory of integration, and the space  $L^p$  of  $p$ -th power integrable functions. These results are used to construct Lebesgue measure in higher dimensions, using Fubini's theorem. Properties of the Hilbert space  $L^2$  are then used to give von Neumann's proof of the Radon–Nikodym theorem, and this is used to establish differentiability properties of measures and functions on  $\mathbf{R}^d$ . Almost all measures that arise in practice are defined on topological spaces, and we establish regularity properties, which show that such measures are rather well behaved. A final chapter uses the theory that we have established to obtain further results, largely concerning Fourier series (first considered in Volume I), and the boundary behaviour of harmonic functions on the unit disc.

The text includes plenty of exercises. Some are straightforward, some are searching, and some contain results needed later. All help develop an understanding of the theory: do them!

I am again extremely grateful to Zhuo Min 'Harold' Lim, who read the proofs and found many errors. Any remaining errors are mine alone. Corrections and further comments can be found on a web page on my personal home page at [www.dpmms.cam.ac.uk](http://www.dpmms.cam.ac.uk).

# Contents

## Volume III

<i>Introduction</i>	<i>page</i>	ix
<b>Part Five Complex analysis</b>		625
<b>20 Holomorphic functions and analytic functions</b>		627
20.1 Holomorphic functions		627
20.2 The Cauchy–Riemann equations		630
20.3 Analytic functions		635
20.4 The exponential, logarithmic and circular functions		641
20.5 Infinite products		645
20.6 The maximum modulus principle		646
<b>21 The topology of the complex plane</b>		650
21.1 Winding numbers		650
21.2 Homotopic closed paths		655
21.3 The Jordan curve theorem		661
21.4 Surrounding a compact connected set		667
21.5 Simply connected sets		670
<b>22 Complex integration</b>		674
22.1 Integration along a path		674
22.2 Approximating path integrals		680
22.3 Cauchy’s theorem		684
22.4 The Cauchy kernel		689
22.5 The winding number as an integral		690
22.6 Cauchy’s integral formula for circular and square paths		692
22.7 Simply connected domains		698
22.8 Liouville’s theorem		699

22.9	Cauchy's theorem revisited	700
22.10	Cycles; Cauchy's integral formula revisited	702
22.11	Functions defined inside a contour	704
22.12	The Schwarz reflection principle	705
<b>23</b>	<b>Zeros and singularities</b>	<b>708</b>
23.1	Zeros	708
23.2	Laurent series	710
23.3	Isolated singularities	713
23.4	Meromorphic functions and the complex sphere	718
23.5	The residue theorem	720
23.6	The principle of the argument	724
23.7	Locating zeros	730
<b>24</b>	<b>The calculus of residues</b>	<b>733</b>
24.1	Calculating residues	733
24.2	Integrals of the form $\int_0^{2\pi} f(\cos t, \sin t) dt$	734
24.3	Integrals of the form $\int_{-\infty}^{\infty} f(x) dx$	736
24.4	Integrals of the form $\int_0^{\infty} x^\alpha f(x) dx$	742
24.5	Integrals of the form $\int_0^{\infty} f(x) dx$	745
<b>25</b>	<b>Conformal transformations</b>	<b>749</b>
25.1	Introduction	749
25.2	Univalent functions on $\mathbf{C}$	750
25.3	Univalent functions on the punctured plane $\mathbf{C}^*$	750
25.4	The Möbius group	751
25.5	The conformal automorphisms of $\mathbf{D}$	758
25.6	Some more conformal transformations	759
25.7	The space $\mathbf{H}(U)$ of holomorphic functions on a domain $U$	763
25.8	The Riemann mapping theorem	765
<b>26</b>	<b>Applications</b>	<b>768</b>
26.1	Jensen's formula	768
26.2	The function $\pi \cot \pi z$	770
26.3	The functions $\pi \operatorname{cosec} \pi z$	772
26.4	Infinite products	775
26.5	*Euler's product formula*	778
26.6	Weierstrass products	783
26.7	The gamma function revisited	790
26.8	Bernoulli numbers, and the evaluation of $\zeta(2k)$	794
26.9	The Riemann zeta function revisited	797

<b>Part Six Measure and Integration</b>	801
<b>27 Lebesgue measure on <math>\mathbf{R}</math></b>	803
27.1 Introduction	803
27.2 The size of open sets, and of closed sets	804
27.3 Inner and outer measure	808
27.4 Lebesgue measurable sets	810
27.5 Lebesgue measure on $\mathbf{R}$	812
27.6 A non-measurable set	814
<b>28 Measurable spaces and measurable functions</b>	817
28.1 Some collections of sets	817
28.2 Borel sets	820
28.3 Measurable real-valued functions	822
28.4 Measure spaces	825
28.5 Null sets and Borel sets	829
28.6 Almost sure convergence	830
<b>29 Integration</b>	834
29.1 Integrating non-negative functions	834
29.2 Integrable functions	839
29.3 Changing measures and changing variables	846
29.4 Convergence in measure	848
29.5 The spaces $L^1_{\mathbf{R}}(X, \Sigma, \mu)$ and $L^1_{\mathbf{C}}(X, \Sigma, \mu)$	854
29.6 The spaces $L^p_{\mathbf{R}}(X, \Sigma, \mu)$ and $L^p_{\mathbf{C}}(X, \Sigma, \mu)$ , for $0 < p < \infty$	856
29.7 The spaces $L^\infty_{\mathbf{R}}(X, \Sigma, \mu)$ and $L^\infty_{\mathbf{C}}(X, \Sigma, \mu)$	863
<b>30 Constructing measures</b>	865
30.1 Outer measures	865
30.2 Caratheodory's extension theorem	868
30.3 Uniqueness	871
30.4 Product measures	873
30.5 Borel measures on $\mathbf{R}$ , $\mathbf{I}$	880
<b>31 Signed measures and complex measures</b>	884
31.1 Signed measures	884
31.2 Complex measures	889
31.3 Functions of bounded variation	891
<b>32 Measures on metric spaces</b>	896
32.1 Borel measures on metric spaces	896
32.2 Tight measures	898
32.3 Radon measures	900

<b>33 Differentiation</b>	903
33.1 The Lebesgue decomposition theorem	903
33.2 Sublinear mappings	906
33.3 The Lebesgue differentiation theorem	908
33.4 Borel measures on $\mathbf{R}$ , II	912
<b>34 Applications</b>	915
34.1 Bernstein polynomials	915
34.2 The dual space of $L^p_{\mathbf{C}}(X, \Sigma, \mu)$ , for $1 \leq p < \infty$	918
34.3 Convolution	919
34.4 Fourier series revisited	924
34.5 The Poisson kernel	927
34.6 Boundary behaviour of harmonic functions	934
<i>Index</i>	936
<i>Contents for Volume I</i>	940
<i>Contents for Volume II</i>	943

# Part Five

Complex analysis



## Holomorphic functions and analytic functions

## 20.1 Holomorphic functions

Suppose that  $f$  is a continuous complex-valued function defined on an open subset  $U$  of the complex plane  $\mathbf{C}$ . Recall that the set  $U$  is the union of countably many connected components, each of which is an open subset of  $U$  (Volume II, Proposition 16.1.15 and Corollary 16.1.18). The behaviour of  $f$  on each component does not depend on its behaviour on the other components. For this reason, we restrict our attention to functions defined on a connected open subset of  $\mathbf{C}$ ; such a set is called a *domain*.

We begin by considering differentiability: the definition is essentially the same as in the real case. Suppose that  $f$  is a complex-valued function on a domain  $U$ , and that  $z \in U$ . Then  $f$  is *differentiable* at  $z$ , with *derivative*  $f'(z)$ , if whenever  $\epsilon > 0$  there exists  $\delta > 0$  such that the open neighbourhood  $N_\delta(z) = \{w : |w - z| < \delta\}$  of  $z$  is contained in  $U$  and such that if  $0 < |w - z| < \delta$  then

$$\left| \frac{f(w) - f(z)}{w - z} - f'(z) \right| < \epsilon.$$

In other words,

$$\frac{f(w) - f(z)}{w - z} \rightarrow f'(z) \text{ as } w \rightarrow z.$$

Thus if  $f$  is differentiable at  $z$ , then the derivative  $f'(z)$  is uniquely determined. The derivative  $f'(z)$  is also denoted by  $\frac{df}{dz}(z)$ .

**Proposition 20.1.1** *Suppose that  $f$  is a complex-valued function on a domain  $U$ , that  $N_\delta(z) \subseteq U$ , and that  $l \in \mathbf{C}$ . The following statements are equivalent.*

- (i)  $f$  is differentiable at  $z$ , with derivative  $l$ .  
 (ii) There is a complex-valued function  $r$  on  $N_\delta^*(0) = N_\delta(0) \setminus \{0\}$  such that

$$f(z+w) = f(z) + lw + r(w) \text{ for } 0 < |w| < \delta$$

for which  $r(w)/w \rightarrow 0$  as  $w \rightarrow 0$ .

- (iii) There is a complex-valued function  $s$  on  $N_\delta(0)$  such that

$$f(z+w) = f(z) + (l + s(w))w \text{ for } |w| < \delta$$

for which  $s(0) = 0$  and  $s$  is continuous at 0.

If so, then  $f$  is continuous at  $z$ .

*Proof* This corresponds to Volume I, Proposition 7.1.1, and the easy proof is essentially the same.  $\square$

If  $f$  is differentiable at every point of  $U$ , then we say that  $f$  is *holomorphic* on  $U$ . If  $U = \mathbf{C}$ , then we say that  $f$  is an *entire* function. Although the form of the definition of differentiability that we have just given is exactly the same as the form of the definition in the real case, we shall see that holomorphic functions are very different from differentiable functions on an open interval of  $\mathbf{R}$ .

**Example 20.1.2** Let  $f(z) = 1/z$  for  $z \in \mathbf{C} \setminus \{0\}$ . Then  $f$  is holomorphic on  $\mathbf{C} \setminus \{0\}$ , with derivative  $-1/z^2$ .

For if  $0 < |w| < |z|$ , then  $z+w \neq 0$  and

$$\frac{f(z+w) - f(z)}{w} - \frac{-1}{z^2} = \frac{z^2 - (z+w)z + w(z+w)}{wz^2(z+w)} = \frac{w}{z^2(z+w)} \rightarrow 0$$

as  $w \rightarrow 0$ .

**Proposition 20.1.3** Suppose that  $f$  and  $g$  are complex-valued functions defined on a domain  $U$ , and that  $f$  and  $g$  are differentiable at  $z$ . Suppose also that  $\lambda, \mu \in \mathbf{C}$ .

- (i)  $\lambda f + \mu g$  is differentiable at  $z$ , with derivative  $\lambda f'(z) + \mu g'(z)$ .  
 (ii) The product  $fg$  is differentiable at  $z$ , with derivative  $f'(z)g(z) + f(z)g'(z)$ .

*Proof* An easy exercise for the reader.  $\square$

**Theorem 20.1.4** (The chain rule) Suppose that  $f$  is a complex-valued function defined on a domain  $U$ , that  $h$  is a complex-valued function defined

on a domain  $V$  and that  $f(U) \subseteq V$ . Suppose that  $f$  is differentiable at  $z$  and that  $h$  is differentiable at  $f(z)$ . Then the composite function  $h \circ f$  is differentiable at  $z$ , with derivative  $h'(f(z)) \cdot f'(z)$ .

*Proof* There are two possibilities. First, there exists  $\delta > 0$  such that  $N_\delta(z) \subseteq U$  and  $f(z+w) \neq f(z)$  for  $0 < |w| < \delta$ . If  $0 < |w| < \delta$  then

$$\frac{h(f(z+w)) - h(f(z))}{w} = \left( \frac{h(f(z+w)) - h(f(z))}{f(z+w) - f(z)} \right) \cdot \left( \frac{f(z+w) - f(z)}{w} \right).$$

Since  $f$  is continuous at  $z$ ,  $f(z+w) - f(z) \rightarrow 0$  as  $w \rightarrow 0$ , and so

$$\frac{h(f(z+w)) - h(f(z))}{f(z+w) - f(z)} \rightarrow h'(f(z)) \text{ as } w \rightarrow 0.$$

Since  $(f(z+w) - f(z))/w \rightarrow f'(z)$  as  $w \rightarrow 0$ , the result follows.

Secondly,  $z$  is the limit point of a sequence  $(z_n)_{n=1}^\infty$  in  $U \setminus \{z\}$  for which  $f(z_n) = f(z)$ . In this case it follows that  $f'(z) = 0$ , and we must show that  $(h \circ f)'(z) = 0$ . We use Proposition 20.1.1. Let  $b = f(z)$ . There exist  $\eta > 0$  such that  $N_\eta(f(z)) \subseteq V$  and a function  $t$  on  $N_\eta(0)$ , with  $t(0) = 0$ , such that  $h(b+k) = h(b) + (h'(b) + t(k))k$  for  $k \in N_\eta(0)$  and such that  $t$  is continuous at 0. Similarly, there exist  $\delta > 0$  such that  $N_\delta(z) \subseteq U$  and a function  $s$  on  $N_\delta(0)$ , with  $s(0) = 0$ , such that  $f(z+w) = b + s(w)w$  for  $w \in N_\delta(0)$  and such that  $s$  is continuous at 0. Since  $f$  is continuous at  $z$ , we can suppose that  $f(N_\delta(z)) \subseteq N_\eta(b)$ . If  $0 < |w| < \delta$  then

$$h(f(z+w)) = h(b + s(w)w) = h(b) + (h'(b) + t(s(w)w))s(w)w$$

so that

$$\frac{h(f(z+w)) - h(f(z))}{w} = (h'(b) + t(s(w)w))s(w) \rightarrow 0 \text{ as } w \rightarrow 0,$$

since  $s(w) \rightarrow 0$  and  $t(s(w)w) \rightarrow 0$  as  $w \rightarrow 0$ . □

This is essentially the same proof as in the real case. But, as we shall see (Theorem 23.1.1), the second case can only arise if  $f$  is constant on  $U$ : complex differentiation is in fact very different from real differentiation.

**Corollary 20.1.5** Suppose that  $g$  is a complex-valued function on  $U$ , which is differentiable at  $z$ . If  $g(z) \neq 0$  then there is a neighbourhood  $N_\delta(z) \subseteq U$  such that  $g(w) \neq 0$  for  $w \in N_\delta(z)$ . The function  $1/g$  on

$N_\delta(z)$  is differentiable at  $z$ , with derivative  $-g'(z)/g(z)^2$ . Furthermore  $f/g$  is differentiable at  $z$ , with derivative

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}.$$

*Proof* Since  $g$  is continuous at  $z$ , there is a neighbourhood  $N_\delta(z) \subseteq U$  such that  $g(w) \neq 0$  for  $w \in N_\delta(z)$ . Then  $g(N_\delta(z)) \subseteq \mathbf{C} \setminus \{0\}$ . Let  $h(z) = 1/z$  for  $z \in \mathbf{C} \setminus \{0\}$ . Then the first result follows from the chain rule, and the second from Proposition 20.1.3.  $\square$

For example, if  $p(z) = a_0 + \cdots + a_n z^n$  is a polynomial function, then  $p$  is an entire function, and  $p'(z) = a_1 + 2a_2 z + \cdots + na_n z^{n-1}$ . Similarly, if  $p$  and  $q$  are polynomials, and  $U$  is an open set in which  $q$  has no zeros then the rational function  $r(z) = p(z)/q(z)$  is holomorphic on  $U$ , and

$$r'(z) = \frac{q(z)p'(z) - q'(z)p(z)}{q(z)^2}.$$

### Exercises

- 20.1.1 Suppose that  $f$  is a holomorphic function on  $N_1(i)$  and that  $(f(z))^5 = z$  for  $z \in N_1(i)$ . What is  $f'(i)$ ?
- 20.1.2 Suppose that  $f$  is a holomorphic function on  $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ . Show that the set  $\{n \in \mathbf{N} : f(1/(n+1)) = 1/n\}$  is finite.

## 20.2 The Cauchy–Riemann equations

Suppose that  $f$  is a complex-valued function on a domain  $U$ , and that  $z = x + iy \in U$ . We can write  $f(z)$  as  $u(x, y) + iv(x, y)$ , where  $u(x, y)$  and  $v(x, y)$  are the real and imaginary parts of  $f(z)$ . The functions  $u$  and  $v$  are real-valued functions of two real variables. How are differentiability properties of  $f$  related to differentiability properties of  $u$  and  $v$ ?

Let us make this more explicit. Let  $k : \mathbf{R}^2 \rightarrow \mathbf{C}$  be defined by setting  $k((x, y)) = x + iy$ ;  $k$  is a linear isometry of  $\mathbf{R}^2$  onto  $\mathbf{C}$ , considered as a real vector space. Let  $j : \mathbf{C} \rightarrow \mathbf{R}^2$  be the inverse mapping. If  $f$  is a complex-valued function on  $U$ , let  $\tilde{f} = j \circ f \circ k$ ;  $\tilde{f}$  is a mapping from the open set  $j(U)$  into  $\mathbf{R}^2$ . If  $\tilde{f}(x, y) = (u(x, y), v(x, y))$ , then  $f(x + iy) = u(x, y) + iv(x, y)$ :

$$\begin{array}{ccc} x + iy & \xrightarrow{f} & f(x + iy) = u(x, y) + iv(x, y) \\ \uparrow k & & \downarrow j \\ (x, y) & \xrightarrow{\tilde{f}} & (u(x, y), v(x, y)) \end{array}$$