D. J. H. GARLING

A COURSE IN Mathematical Analysis

VOLUME III

Complex Analysis, Measure and Integration

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A COURSE IN MATHEMATICAL ANALYSIS

Volume III
Complex Analysis, Measure and Integration

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A COURSE IN MATHEMATICAL ANALYSIS

Volume III: Complex Analysis, Measure and Integration

The three volumes of *A Course in Mathematical Analysis* provide a full and detailed account of all those elements of real and complex analysis that an undergraduate mathematics student can expect to encounter in the first two or three years of study. Containing hundreds of exercises, examples and applications, these books will become an invaluable resource for both students and instructors.

Volume I focuses on the analysis of real-valued functions of a real variable. Volume II goes on to consider metric and topological spaces, and functions of a vector variable, and includes an introduction to the theory of manifolds in Euclidean space. This third volume develops the classical theory of functions of a complex variable. It carefully establishes the properties of the complex plane, including a proof of the Jordan curve theorem. Lebesgue measure is introduced, and is used as a model for other measure spaces, where the theory of integration is developed. The Radon–Nikodym theorem is proved, and the differentiation of measures is discussed.

D. J. H. GARLING is Emeritus Reader in Mathematical Analysis at the University of Cambridge and Fellow of St. John's College, Cambridge. He has fifty years' experience of teaching undergraduate students in most areas of pure mathematics, but particularly in analysis.

Introduction

This book is the third and final volume of a full and detailed course in the elements of real and complex analysis that mathematical undergraduates may expect to meet. Indeed, I have based it on those parts of analysis that undergraduates at Cambridge University meet, or used to meet, in their first two years. I have however found it desirable to go rather further in certain places, in order to give a rounded account of the material.

In Part Five, we develop the theory of functions of a complex variable. To begin with, we consider holomorphic functions (functions which are complex-differentiable) and analytic functions (functions which can be defined by power series), and the results seem similar to those of real case. Things change when path-integrals are introduced. To use these, a good understanding of the topology of the plane is needed. We give a careful account of this, including a proof of the Jordan curve theorem (every simple closed curve has an inside and an outside). With this in place, various forms of Cauchy's theorem and Cauchy's integral formula are proved. These lead on to many magical results. Chapter 25 is geometric. A single-valued holomorphic function is conformal (that is, it preserves angles and orientations). We consider the problem of mapping one domain conformally onto another, and end by proving the celebrated Riemann mapping theorem, which says that if U and V are domains in the complex plane which are proper subsets of the plane and are simply-connected (there are no holes) then there exists a conformal mapping of U onto V. In Chapter 26, we apply the theory that we have developed to various problems, some of which were first introduced in Volume I.

In Volume I, we developed properties of the Riemann integral. This is very satisfactory when we wish to integrate continuous or monotonic functions, and is a useful precursor for the complex path integrals that we consider in Part Five, but it has serious shortcomings. In Part Six, we introduce

Lebesgue measure on the real line. Abstract measure theory is a large and important subject, but the topological properties of the real line make the construction of Lebesgue measure on the real line rather straightforward. With this example in place, we introduce the notion of a measure space, and the corresponding space of measurable functions. This then leads on easily to the theory of integration, and the space L^p of p-th power integrable functions. These results are used to construct Lebesgue measure in higher dimensions, using Fubini's theorem. Properties of the Hilbert space L^2 are then used to give von Neumann's proof of the Radon–Nikodym theorem, and this is used to establish differentiability properties of measures and functions on $\mathbf{R}^d$. Almost all measures that arise in practice are defined on topological spaces, and we establish regularity properties, which show that such measures are rather well behaved. A final chapter uses the theory that we have established to obtain further results, largely concerning Fourier series (first considered in Volume I), and the boundary behaviour of harmonic functions on the unit disc.

The text includes plenty of exercises. Some are straightforward, some are searching, and some contain results needed later. All help develop an understanding of the theory: do them!

I am again extremely grateful to Zhuo Min 'Harold' Lim, who read the proofs and found many errors. Any remaining errors are mine alone. Corrections and further comments can be found on a web page on my personal home page at `www.dpmms.cam.ac.uk`.

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Holomorphic functions and analytic functions

20.1 Holomorphic functions

Suppose that f is a continuous complex-valued function defined on an open subset U of the complex plane $\mathbf{C}$. Recall that the set U is the union of countably many connected components, each of which is an open subset of U (Volume II, Proposition 16.1.15 and Corollary 16.1.18). The behaviour of f on each component does not depend on its behaviour on the other components. For this reason, we restrict our attention to functions defined on a connected open subset of $\mathbf{C}$; such a set is called a *domain*.

We begin by considering differentiability: the definition is essentially the same as in the real case. Suppose that f is a complex-valued function on a domain U, and that $z \in U$. Then f is *differentiable* at z, with *derivative* $f'(z)$, if whenever $\epsilon > 0$ there exists $\delta > 0$ such that the open neighbourhood $N_\delta(z) = \{w : |w - z| < \delta\}$ of z is contained in U and such that if $0 < |w - z| < \delta$ then

$$\left| \frac{f(w) - f(z)}{w - z} - f'(z) \right| < \epsilon.$$

In other words,

$$\frac{f(w) - f(z)}{w - z} \to f'(z) \text{ as } w \to z.$$

Thus if f is differentiable at z, then the derivative $f'(z)$ is uniquely determined. The derivative $f'(z)$ is also denoted by $\frac{df}{dz}(z)$.

Proposition 20.1.1 *Suppose that f is a complex-valued function on a domain U, that $N_\delta(z) \subseteq U$, and that $l \in \mathbf{C}$. The following statements are equivalent.*

(i) f is differentiable at z, with derivative l.

(ii) There is a complex-valued function r on $N^*_\delta(0) = N_\delta(0) \setminus \{0\}$ such that

$$f(z+w) = f(z) + lw + r(w) \text{ for } 0 < |w| < \delta$$

for which $r(w)/w \to 0$ as $w \to 0$.

(iii) There is a complex-valued function s on $N_\delta(0)$ such that

$$f(z+w) = f(z) + (l + s(w))w \text{ for } |w| < \delta$$

for which $s(0) = 0$ and s is continuous at 0.

If so, then f is continuous at z.

Proof This corresponds to Volume I, Proposition 7.1.1, and the easy proof is essentially the same. □

If f is differentiable at every point of U, then we say that f is *holomorphic* on U. If $U = \mathbf{C}$, then we say that f is an *entire* function. Although the form of the definition of differentiability that we have just given is exactly the same as the form of the definition in the real case, we shall see that holomorphic functions are very different from differentiable functions on an open interval of $\mathbf{R}$.

Example 20.1.2 Let $f(z) = 1/z$ for $z \in \mathbf{C} \setminus \{0\}$. Then f is holomorphic on $\mathbf{C} \setminus \{0\}$, with derivative $-1/z^2$.

For if $0 < |w| < |z|$, then $z + w \neq 0$ and

$$\frac{f(z+w) - f(z)}{w} - \frac{-1}{z^2} = \frac{z^2 - (z+w)z + w(z+w)}{wz^2(z+w)} = \frac{w}{z^2(z+w)} \to 0$$

as $w \to 0$.

Proposition 20.1.3 *Suppose that f and g are complex-valued functions defined on a domain U, and that f and g are differentiable at z. Suppose also that $\lambda, \mu \in \mathbf{C}$.*

(i) $\lambda f + \mu g$ is differentiable at z, with derivative $\lambda f'(z) + \mu g'(z)$.

(ii) The product fg is differentiable at z, with derivative $f'(z)g(z) + f(z)g'(z)$.

Proof An easy exercise for the reader. □

Theorem 20.1.4 (The chain rule) *Suppose that f is a complex-valued function defined on a domain U, that h is a complex-valued function defined* *on a domain V and that $f(U) \subseteq V$. Suppose that f is differentiable at z and that h is differentiable at $f(z)$. Then the composite function $h \circ f$ is differentiable at z, with derivative $h'(f(z)).f'(z)$.*

Proof There are two possibilities. First, there exists $\delta > 0$ such that $N_\delta(z) \subseteq U$ and $f(z+w) \neq f(z)$ for $0 < |w| < \delta$. If $0 < |w| < \delta$ then

$$\frac{h(f(z+w)) - h(f(z))}{w} = \left(\frac{h(f(z+w)) - h(f(z))}{f(z+w) - f(w)}\right) \cdot \left(\frac{f(z+w) - f(z)}{w}\right).$$

Since f is continuous at z, $f(z+w) - f(z) \to 0$ as $w \to 0$, and so

$$\frac{h(f(z+w)) - h(f(z))}{f(z+w) - f(z)} \to h'(f(z)) \text{ as } w \to 0.$$

Since $(f(z+w) - f(z))/w \to f'(z)$ as $w \to 0$, the result follows.

Secondly, z is the limit point of a sequence $(z_n)_{n=1}^\infty$ in $U \setminus \{z\}$ for which $f(z_n) = f(z)$. In this case it follows that $f'(z) = 0$, and we must show that $(h \circ f)'(z) = 0$. We use Proposition 20.1.1. Let $b = f(z)$. There exist $\eta > 0$ such that $N_\eta(f(z)) \subseteq V$ and a function t on $N_\eta(0)$, with $t(0) = 0$, such that $h(b+k) = h(b) + (h'(b) + t(k))k$ for $k \in N_\eta(0)$ and such that t is continuous at 0. Similarly, there exist $\delta > 0$ such that $N_\delta(z) \subseteq U$ and a function s on $N_\delta(0)$, with $s(0) = 0$, such that $f(z+w) = b + s(w)w$ for $h \in N_\delta(0)$ and such that s is continuous at 0. Since f is continuous at z, we can suppose that $f(N_\delta(z)) \subseteq N_\eta(b)$. If $0 < |w| < \delta$ then

$$h(f(z+w)) = h(b + s(w)w) = h(b) + (h'(b) + t(s(w)w))s(w)w$$

so that

$$\frac{h(f(z+w)) - h(f(z))}{w} = (h'(b) + t(s(w)w))s(w) \to 0 \text{ as } w \to 0,$$

since $s(w) \to 0$ and $t(s(w)w) \to 0$ as $w \to 0$. □

This is essentially the same proof as in the real case. But, as we shall see (Theorem 23.1.1), the second case can only arise if f is constant on U: complex differentiation is in fact very different from real differentiation.

Corollary 20.1.5 *Suppose that g is a complex-valued function on U, which is differentiable at z. If $g(z) \neq 0$ then there is a neighbourhood $N_\delta(z) \subseteq U$ such that $g(w) \neq 0$ for $w \in N_\delta(z)$. The function $1/g$ on*

$N_\delta(z)$ is differentiable at z, with derivative $-g'(z)/g(z)^2$. Furthermore f/g is differentiable at z, with derivative

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}.$$

Proof Since g is continuous at z, there is a neighbourhood $N_\delta(z) \subseteq U$ such that $g(w) \neq 0$ for $w \in N_\delta(z)$. Then $g(N_\delta(z)) \subseteq \mathbf{C} \setminus \{0\}$. Let $h(z) = 1/z$ for $z \in \mathbf{C} \setminus \{0\}$. Then the first result follows from the chain rule, and the second from Proposition 20.1.3. □

For example, if $p(z) = a_0 + \cdots + a^n z^n$ is a polynomial function, then p is an entire function, and $p'(z) = a_1 + 2a_2 z + \cdots + na_n z^{n-1}$. Similarly, if p and q are polynomials, and U is an open set in which q has no zeros then the rational function $r(z) = p(z)/q(z)$ is holomorphic on U, and

$$r'(z) = \frac{q(z)p'(z) - q'(z)p(z)}{q(z)^2}.$$

Exercises

20.1.1 Suppose that f is a holomorphic function on $N_1(i)$ and that $(f(z))^5 = z$ for $z \in N_1(i)$. What is $f'(i)$?

20.1.2 Suppose that f is a holomorphic function on $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$. Show that the set $\{n \in \mathbf{N} : f(1/(n+1)) = 1/n\}$ is finite.

20.2 The Cauchy–Riemann equations

Suppose that f is a complex-valued function on a domain U, and that $z = x + iy \in U$. We can write $f(z)$ as $u(x,y) + iv(x,y)$, where $u(x,y)$ and $v(x,y)$ are the real and imaginary parts of $f(z)$. The functions u and v are real-valued functions of two real variables. How are differentiability properties of f related to differentiability properties of u and v?

Let us make this more explicit. Let $k : \mathbf{R}^2 \to \mathbf{C}$ be defined by setting $k((x,y)) = x + iy$; k is a linear isometry of $\mathbf{R}^2$ onto $\mathbf{C}$, considered as a real vector space. Let $j : \mathbf{C} \to \mathbf{R}^2$ be the inverse mapping. If f is a complex-valued function on U, let $\widetilde{f} = j \circ f \circ k$; $\widetilde{f}$ is a mapping from the open set $j(U)$ into $\mathbf{R}^2$. If $\widetilde{f}(x,y) = (u(x,y), v(x,y))$, then $f(x+iy) = u(x,y) + iv(x,y)$:

$$\begin{array}{ccc} x + iy & \xrightarrow{f} & f(x+iy) = u(x,y) + iv(x,y) \\ k \big\uparrow & & \big\downarrow j \\ (x,y) & \xrightarrow{\widetilde{f}} & (u(x,y), v(x,y)) \end{array}$$