TABLES OF THE FUNCTION arc sin z

BY

THE STAFF OF THE COMPUTATION LABORATORY

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INTRODUCTION

I

THE INVERSE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS IN THE COMPLEX DOMAIN

The MacLaurin series,

$$w - \frac{w^3}{3!} + \frac{w^5}{5!} - \cdots,$$

is convergent for all complex numbers w, and so defines in the complex plane a function identical with the sine when the argument is real. Hence the series serves to extend to the complex domain the elementary function $x = \sin u$. Many of the well-known properties of the real function persist in the complex domain. Thus, defining

$$\cos w = \frac{d}{dw} \sin w = 1 - \frac{w^2}{2!} + \frac{w^4}{4!} - \cdots,$$

one may readily extend the addition formula

$$\sin (w_1+w_2) = \sin w_1 \cos w_2 + \cos w_1 \sin w_2, \qquad (1)$$

to the case of any pair of complex numbers w_1 and w_2. The extension is made in two steps by application of the principle of permanence of functional relations, or by algebraic manipulations of the series expressions. Similarly, the Pythagorean relation, $\sin^2 w + \cos^2 w = 1$, is valid for all complex numbers w.

One may define the hyperbolic sine and cosine as

$$\sinh w = -i \sin iw, \qquad \cosh w = \cos iw,$$

and observe by use of the series that $\sinh v$ and $\cosh v$ are real-valued functions of a real variable v. Taking $w_1 = u$, $w_2 = vi$ in the addition formula one has

$$\sin (u+vi) = \sin u \cosh v + i \cos u \sinh v. \qquad (2)$$

This formula provides a device for evaluating the real and imaginary parts of the function $\sin w$. The device is so easy to apply that it would seem unnecessary to tabulate the function $\sin w$, but the function is of such fundamental importance that a number of tables have been made. Some indication of the usefulness of such tables is given by the fact that three editions of Kennelly's *Chart Atlas* appeared between 1914 and 1924.[1]

In the case of the inverse function,

$$w = \text{arc sin } z,$$

the real and imaginary parts of w can likewise be obtained by means of formulas whose evaluation involves only elementary real functions. However, the formulas are not nearly so simple to apply, since they involve square roots, together with inverse circular and hyperbolic functions; in addition, the presence of multiple-valued functions in the formulas tends to provide disagreeable ambiguities. Despite these difficulties there has been very little tabulation of the inverse sine. In the only other tables known to the present writers the argument is in polar coordinates,[2,3] while in the present tables the argument and function are both in Cartesian form.

The properties of the inverse function will be studied by means of conformal mapping. As a preliminary, certain properties of the function sin w will be listed. The real-valued sine function is odd, and has period 2π. The values of the second half-period reflect the values of the first half-period, according to the law,

$$\sin(\pi+x) = -\sin x,$$

and the values of the second quarter-period reflect the values of the first according to the law,

$$\sin(\tfrac{1}{2}\pi+x) = \sin(\tfrac{1}{2}\pi-x) = \cos x.$$

Hence in the real domain a tabulation of the first quarter-period suffices to tabulate the sine for all real values, and the cosine as well. All of these properties extend to the complex domain, and indeed may be verified at once by means of the addition formula. Note also that if

then
$$\left.\begin{aligned} x+yi &= \sin(u+vi), \\ -x+yi &= \sin(-u+vi), \\ x-yi &= \sin(u-vi), \\ -x-yi &= \sin(-u-vi); \end{aligned}\right\} \qquad (3)$$

these relations are an immediate consequence of the addition law. By the relations (3), the sine is known everywhere if it is known in the first quadrant. By the mirror-periodicity on π, the sine is known everywhere if it is known in the strip $0 \leq x < \pi$. Combining these, the sine is known everywhere if it is known in the half-strip $0 \leq x < \pi$, $0 \leq y$.

The function $z = \sin w$ defines a mapping of the w-plane onto the z-plane. The origin is mapped into the origin, the interval $(0, \frac{1}{2}\pi)$ of the u-axis is mapped into the interval $(0, 1)$ of the x-axis in a one-to-one manner. The point $(\frac{1}{2}\pi, v)$ in the w-plane goes into the point $(\cosh v, 0)$ in the z-plane. Accordingly, the upper half of the line $u = \frac{1}{2}\pi$ is mapped into the portion $x \geq 1$ of the x-axis in a one-to-one manner. The point $(0, v)$ goes into the point $(0, \sinh v)$, thereby establishing a one-to-one mapping of the positive part of the v-axis into the positive part of the y-axis. By continuity, the region bounded by the lines $u = 0$, $u = \frac{1}{2}\pi$, and the u-axis must then be carried into the first quadrant of the z-plane. In Fig. 1, the quarter-strip w_{I} is mapped into the quadrant z_{I}. It will presently be verified that this

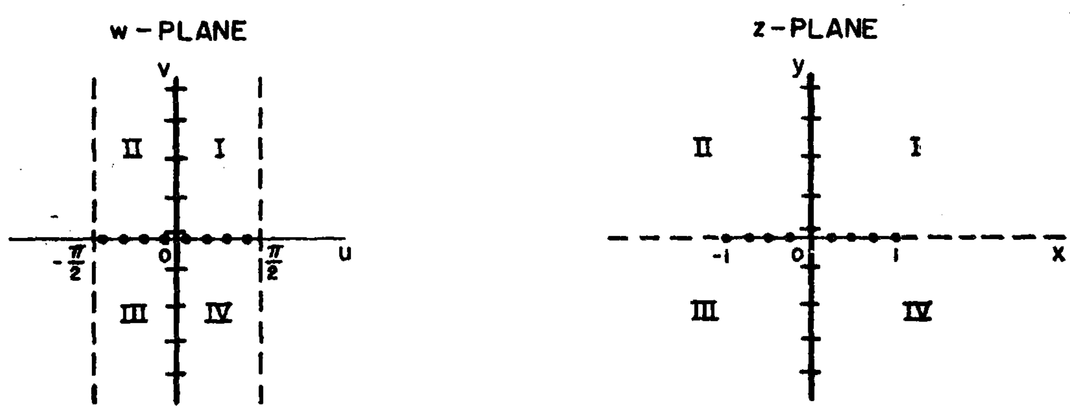

Fig. 1

mapping is entirely one-to-one. It follows from the reflectivity relations (3) that the quarter-strips w_{II}, w_{III}, and w_{IV} are mapped into the quadrants z_{II}, z_{III}, and z_{IV} respectively. Assuming that these four mappings are each one-to-one throughout the interiors and on the boundaries, it is seen that the mapping of the strip $-\frac{1}{2}\pi \leq u \leq \frac{1}{2}\pi$ onto the z-plane is one-to-one at all points not on the x-axis, and in fact fails to be one-to-one only at points of the x-axis exterior to the unit circle. Each such point is derived from two points, since the point $(\cosh v, 0)$ has as antecedents the points $(\frac{1}{2}\pi, \pm v)$, and $(-\cosh v, 0)$ has as antecedents the points $(-\frac{1}{2}\pi, \pm v)$.

According to (2) the point (u, v) is mapped into the point (x, y),

$$x = \sin u \cosh v, \qquad y = \cos u \sinh v, \tag{4}$$

and is uniquely determined. From (4),

$$x^2 \cos^2 u - y^2 \sin^2 u = \cos^2 u \sin^2 u, \tag{4a}$$

$$x^2 \sinh^2 v + y^2 \cosh^2 v = \sinh^2 v \cosh^2 v. \tag{4b}$$

If u is held constant (4a) represents a hyperbola. This means that points of the line $u = c$ are mapped into points of the hyperbola (4a). Since the quadrant in which a point falls is preserved by the mapping, the upper and lower halves of the line $u = c$ are mapped into the upper and lower halves of a branch of the hyperbola (4a); the remaining branch is provided as the transform of the line $u = -c$. Similarly the pair of lines $v = \pm c$ map into the ellipse (4b) (see Fig. 2). It follows from

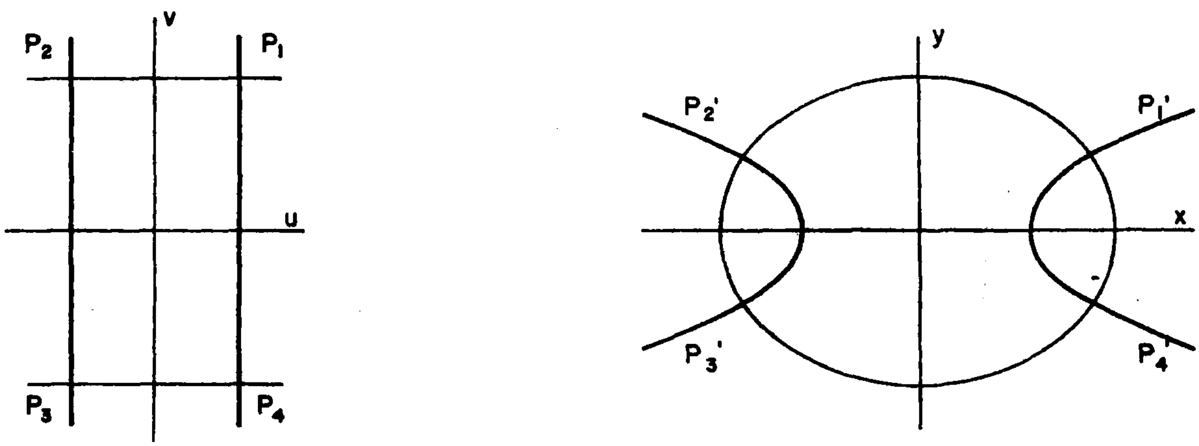

Fig. 2

continuity that the entire ellipse and hyperbola are generated, and not merely portions of them. Thus the family of lines parallel to the v-axis maps into the family of hyperbolas (4a), and the family of lines parallel to the u-axis maps into the family of ellipses (4b); since the mapping is conformal the families (4a) and (4b) are mutually orthogonal. A given point (x, y) is determined as a point of intersection of a uniquely determined pair of orthogonal conics whose antecedent is a pair of straight lines. The unique point of intersection of the latter is the unique antecedent of the point (x, y).

If k is a rational integer the locus of points satisfying

$$(k-\tfrac{1}{2})\pi \leq u \leq (k+\tfrac{1}{2})\pi$$

is called the *kth period strip*. Every period strip is mapped into the entire z-plane. If k is even this follows from the periodicity

$$\sin(2n\pi+w) = \sin w,$$

and if k is odd this follows from the relation

$$\sin\left((2n+1)\pi+w\right) = -\sin w.$$

The mapping of the kth period strip into the z-plane is like that of the zeroth if k is even, and is its mirror-image in the axis of imaginaries if k is odd. The correspondence between quarter-strips and quadrants is indicated in Fig. 3.

The inverse function, $w = \arcsin z$, may now be defined by inverting the mapping. To each point $z = x+yi$ there corresponds a point $w = u+vi$, and with it all points $(w+2k\pi)$, $(2k+1)\pi-w$, where k is a rational integer.

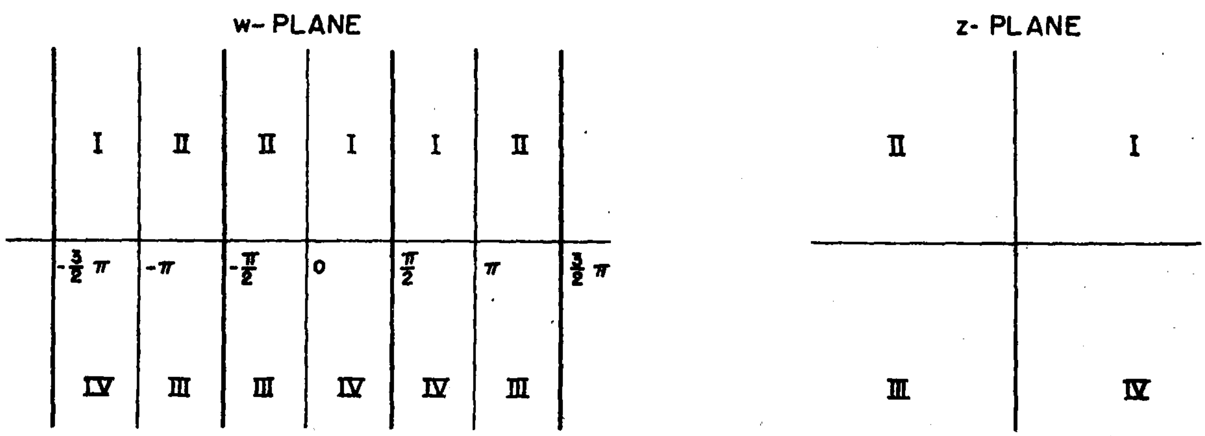

Fig. 3

In any period strip the point w is unique, except when z is on the x-axis and outside the unit circle, in which case there are two points w, corresponding to $w = u\pm vi$. It is convenient for some purposes to define a *principal value* of the inverse sine; this will be done here by choosing the values w that lie in the interior of the period strip $-\frac{1}{2}\pi \leq u \leq \frac{1}{2}\pi$ and including with them the boundary points that lie above the axis of reals. This definition is an extension of the one used in the real domain, and consistent with it. The notation $\arcsin z$ will hereafter refer to the principal value, except where it is expressly indicated otherwise. It may be noted at once that the symmetries of the sine are valid for its inverse, that is, if

then
$$\left.\begin{aligned} u+vi &= \text{arc sin } (x+yi), \\ -u+vi &= \text{arc sin } (-x+yi), \\ u-vi &= \text{arc sin } (x-yi), \\ -u-vi &= \text{arc sin } (-x-yi). \end{aligned}\right\} \quad (5)$$

Accordingly, it is necessary to tabulate the first quadrant only.

As mentioned previously, given x and y, the corresponding values u and v may be computed directly. In fact, letting $r = \sin^2 u$ and $b = x^2+y^2+1$, one may write (4a) as a quadratic equation in r,

$$r^2 - br + x^2 = 0. \quad (6)$$

Similarly, the substitution $r = \cosh^2 v$ transforms (4b) into a quadratic equation, which turns out to be identical with (6). Hence $\sin^2 u$ and $\cosh^2 v$ are the roots

$$\tfrac{1}{2}(b \pm \sqrt{b^2-4x^2}) = \tfrac{1}{2}(b \pm D)$$

of (6). Since $\sin^2 u \leq 1 \leq \cosh^2 v$, it follows that $2 \sin^2 u = b-D$, $2 \cosh^2 v = b+D$, where

$$D = \sqrt{b^2-4x^2} = \sqrt{(x^2+y^2)^2 + 1 - 2(x^2-y^2)}. \quad (7)$$

Next one may write

$$\left.\begin{aligned} c &= \cos 2u = 1 - 2 \sin^2 u = 1 - (b-D) = -(x^2+y^2) + D, \\ d &= \cosh 2v = 2 \cosh^2 v - 1 = (b+D) - 1 = (x^2+y^2) + D. \end{aligned}\right\} \quad (8)$$

The expressions (8) lead to the required expressions for u and v in the form

$$u = \tfrac{1}{2} \text{ arc cos } c, \qquad v = \tfrac{1}{2} \text{ arc cosh } d. \quad (9)$$

In passing, let the identities

$$(1-c)(1+d) = 4x^2, \qquad (1+c)(d-1) = 4y^2, \quad (10a)$$

$$(1+c)(1+d) = 2(1-x^2+y^2+D), \qquad (1-c)(d-1) = 2(-1+x^2-y^2+D), \quad (10b)$$

be noted for later use.

When the formulas (9) are used any apparent ambiguities may be resolved according to standard usage; thus D is the positive square root, the number arc cosh d is real and non-negative, the number arc cos c is the ordinary principal value, a number between 0 and π satisfying the relations

$$\text{arc cos } (-c) + \text{arc cos } c = \pi, \qquad \text{arc sin } c + \text{arc cos } c = \tfrac{1}{2}\pi.$$

This determines $|u|$ and $|v|$, after which algebraic signs are adjoined so as to place $u+vi$ in the same quadrant as $x+yi$. The number $u+vi$ thereby obtained is the principal value w of arc sin z, from which all other values can be obtained as the numbers $2k\pi+w$, $(2k+1)\pi-w$, where k is a rational integer.

When x or y is large the formulas (7) and (8) are less convenient to apply. Notice, for example, that the first of Eqs. (8) expresses c as the difference of two large, nearly equal quantities. In such cases the binomial expansion becomes very useful. A few developments based on it will be obtained at once.

Let $G = x^2+y^2$, $F = x^2-y^2$, $L = (1-2F)/G^2$. Then

$$D = (G^2+1-2F)^{\frac{1}{2}} = G\left[1 + \frac{1}{2}L - \frac{1}{8}L^2 + \frac{1}{16}L^3 - \cdots\right],$$

$$c = -G + D = G\left[\frac{1}{2}L - \frac{1}{8}L^2 + \frac{1}{16}L^3 - \cdots\right]$$

$$= -\frac{F}{G} + \frac{1}{2}\,\frac{G^2-F^2}{G^3} + \frac{1}{2}\,\frac{F(G^2-F^2)}{G^5} + \cdots = -\frac{x^2-y^2}{x^2+y^2} + \rho,$$

$$\rho = \frac{2x^2y^2}{(x^2-y^2)^3}\left[1 + \frac{x^2-y^2}{(x^2+y^2)^2} + \cdots\right].$$

Since the relation arc cos $c = \frac{1}{2}\pi$ - arc sin c holds whether c is positive or negative, one may write

$$u = \tfrac{1}{2}\text{ arc cos } c = \tfrac{1}{4}\pi - \tfrac{1}{2}\text{ arc sin}\left(-\frac{x^2-y^2}{x^2+y^2} + \rho\right). \tag{11a}$$

Next

$$d = G + D = G\left[2 + \frac{1}{2}L - \frac{1}{8}L^2 + \cdots\right],$$

$$d + \sqrt{d^2-1} = 2d - \frac{1}{2d} - \frac{1}{8d^3} - \cdots$$

$$= G(4+L-\tfrac{1}{4}L^2+\cdots) - \frac{1}{G(4+L-\frac{1}{4}L^2+\cdots)} - \frac{1}{G^3(4+L-\frac{1}{4}L^2+\cdots)^3} + \cdots$$

$$= 4G + \frac{(3-8F)}{4G} - \frac{F^2}{G^3} + \cdots = 4(x^2+y^2) + \rho',$$

$$\rho' = \frac{(3-8x^2+8y^2)}{4(x^2+y^2)} - \frac{(x^2-y^2)^2}{(x^2+y^2)^3} + \cdots .$$

Hence

$$v = \tfrac{1}{2}\text{ arc cosh } d = \tfrac{1}{2}\log_e\,(d + \sqrt{d^2-1}) = \tfrac{1}{2}\log_e\left[4(x^2+y^2) + \rho'\right]. \qquad (11b)$$

Again,

$$\left.\begin{aligned}\frac{1}{D} &= (G^2+1-2F)^{-\frac{1}{2}} = G^{-1}(1 - \tfrac{1}{2}L + \tfrac{3}{8}L^2 - \tfrac{5}{16}L^3 + \cdots)\\ &= G^{-1}(1 + \frac{F}{G^2} + \frac{3F^2-G^2}{2G^4} + \frac{5F^3-3FG^2}{2G^6} + \cdots)\\ &= \frac{1}{G} + \frac{r}{G^2} + \frac{3r^2-1}{2G^3} + \frac{5r^3-3r}{2G^4} + \cdots,\end{aligned}\right\} \qquad (12a)$$

where $r = F/G = (x^2-y^2)/(x^2+y^2)$. Similarly,

$$\left.\begin{aligned}\frac{1-F}{D^2} &= \frac{1-F}{G^2-2F+1} = \frac{(1-F)(1+L)^{-1}}{G^2} = \frac{(-F+1)(1-L+L^2-L^3+\cdots)}{G^2}\\ &= -\frac{F}{G^2} - \frac{F(2F-1)}{G^4} - \frac{F(2F-1)^2}{G^6} - \frac{F(2F-1)^3}{G^8} - \cdots\\ &\quad + \frac{1}{G^2} + \frac{2F-1}{G^4} + \frac{(2F-1)^2}{G^6} + \cdots\\ &= -\frac{r}{G} + \frac{1-2r^2}{G^2} + \frac{3r-4r^3}{G^3} + \frac{-1+8r^2-8r^4}{G^4} + \cdots .\end{aligned}\right\} \qquad (12b)$$

The last two expansions have been obtained in anticipation of the discussion of the derivative of the arc sine, which is considered next.

The derivative of w = arc sin z is expressed formally by

$$\frac{dw}{dz} = \frac{1}{\sqrt{1-z^2}},$$

and is in fact that branch of this function that coincides with $1/\sqrt{1-x^2}$ for small real x. The function arc sin z has a finite derivative at all complex points except at the branch points $z = \pm 1$. The derivative dw/dz will be denoted more briefly by w', and its respective real and imaginary parts by u' and v'. More generally, the nth derivative $d^n w/dz^n$ will be denoted by $w^{(n)}$, and its real and imaginary parts by $u^{(n)}$ and $v^{(n)}$ respectively.

It has just been noted that $w' = 1/\sqrt{1 - (x+yi)^2}$. By squaring and rationalizing this may be separated into its real and imaginary parts

$$u'^2 - v'^2 = \frac{(1-x^2+y^2)}{\Delta}, \qquad u'v' = \frac{xy}{\Delta}, \tag{13a}$$

where Δ is a function of x and y, defined by

$$\Delta = D^2 = (x^2+y^2)^2 + 1 - 2(x^2-y^2).$$

The simultaneous quadratic equations (13*a*) can be solved simultaneously to give

$$u' = s_1\sqrt{\frac{1-x^2+y^2+s_2D}{2\Delta}}, \qquad v' = s_1\sqrt{\frac{-1+x^2-y^2+s_2D}{2\Delta}}. \tag{13b}$$

In (13*b*), $s_1^2 = s_2^2 = 1$. Actually, $s_1 = s_2 = 1$; instead of verifying this directly it will be obtained through the use of alternate expressions for u' and v' obtained by means of the Cauchy-Riemann differential equations. Thus, from (8) and (9),

$$u' = \frac{\partial u}{\partial x} = \frac{x}{D}\sqrt{\frac{1+c}{1-c}}, \qquad v' = \frac{\partial v}{\partial x} = \frac{x}{D}\sqrt{\frac{d-1}{d+1}},$$

on the one hand, and on the other,

$$u' = \frac{\partial v}{\partial y} = \frac{y}{D}\sqrt{\frac{d+1}{d-1}}, \qquad v' = -\frac{\partial u}{\partial y} = \frac{y}{D}\sqrt{\frac{1-c}{1+c}}.$$

Three different sets of analytic expressions have been obtained for u', v'. Their equivalence may be verified by use of the identities (10*a*) and (10*b*), but only the choice $s_1 = s_2 = 1$ will reconcile a Cauchy-Riemann expression for the derivative with the one in (13*b*). The expansions (12*a*) and (12*b*) can be used to evaluate $2u'^2$ and $2v'^2$ when x or y is large.

The second derivative is $w'' = z(1-z^2)^{-3/2} = zw'^3$. More generally, for n exceeding unity,

$$w^{(n)} = G_n w'^2, \qquad G_n = (2n-3)w^{(n-1)}z + (n-2)^2 w^{(n-2)}.$$

This can be proved by induction. It is true for $n = 2$. Assume it is true for n. Notice that $G_n w'' = G_n w'^3 z = w^{(n)} w' z$. The derivative of order $n+1$ is

$$w^{(n+1)} = w'^2 \frac{dG_n}{dz} + 2G_n w' w''.$$

On insertion of the expanded expressions for G_n and its derivative this becomes

$$w^{(n+1)} = w'^2\left[(2n-3)w^{(n)}z + (2n-3)w^{(n-1)} + (n-2)^2 w^{(n-1)}\right] + 2w'^2 w^{(n)}z$$
$$= \left[(2n-1)w^{(n)}z + (n-1)^2 w^{(n-1)}\right]w'^2 = G_{n+1}w'^2,$$

completing the induction. The real and imaginary parts of the recurrence formula for $w^{(n)}$ may be written separately as

$$\begin{aligned} u^{(n)} &= \left[(2n-3)(xu^{(n-1)}-yv^{(n-1)}) + (n-2)^2u^{(n-2)}\right](u'^2-v'^2) \\ &\quad - \left[(2n-3)(xv^{(n-1)}+yu^{(n-1)}) + (n-2)^2v^{(n-2)}\right]2u'v', \\ v^{(n)} &= \left[(2n-3)(xv^{(n-1)}+yu^{(n-1)}) + (n-2)^2v^{(n-2)}\right](u'^2-v'^2) \\ &\quad + \left[(2n-3)(xu^{(n-1)}-yv^{(n-1)}) + (n-2)^2u^{(n-2)}\right]2u'v'. \end{aligned} \tag{14}$$

The function $w = \text{arc sin } z$ may be expanded in Taylor's series about any point of the complex plane, with the exception of the branch points $z = \pm 1$; and the coefficients of the expansion may be calculated with the aid of the expressions for the derivatives that have been obtained here. At the branch point $z = 1$ the function is not analytic, but the function

$$G(z) = \frac{(\frac{1}{2}\pi - \text{arc sin } z)}{(2-2z)^{\frac{1}{2}}} = \frac{\text{arc cos } z}{(2-2z)^{\frac{1}{2}}}$$

is analytic there, and may be expanded in Taylor's series. The values $G(1), G'(1), G''(1), \ldots,$ may be calculated with the aid of l'Hôpital's rule, and this leads to the series

$$G(z) = 1 + \frac{1}{3\cdot 4}(1-z) + \frac{1\cdot 3}{2!5\cdot 4^2}(1-z)^2 + \frac{1\cdot 3\cdot 5}{3!7\cdot 4^3}(1-z)^3 + \cdots .$$

It may be seen that this series converges in the interior of a circle of radius two and center unity; hence the function arc cos z is represented there by the function

$$\text{arc cos } z = (2-2z)^{\frac{1}{2}}G(z). \tag{15a}$$

From the expansion (15a) may be obtained a corresponding expansion, converging at interior points of a circle of radius two and center $z = -1$:

$$\text{arc cos } z = \pi - \text{arc cos }(-z) = \pi - (2+2z)^{\frac{1}{2}}G(-z). \tag{15b}$$

For practical purposes it is desirable to separate the real and imaginary parts of (15a). To avoid a tedious calculation in this separation one may proceed as follows. The real part is, by (9), $u = \frac{1}{2}$ arc cos c. If z is near 1 then c is near -1 and arc cos c may be calculated by use of (15b):

$$u = \tfrac{1}{2}\left[\pi - (2+2c)^{\frac{1}{2}}G(-c)\right].$$

An expression for the imaginary part may be found in much the same manner. By (9), $v = \frac{1}{2}$ arc cosh $d = -\frac{1}{2}i$ arc cos d. If z is near 1 then so is d; moreover, the real number $1-d$ is negative. Hence, using (15a),

$$v = -\tfrac{1}{2}i(2-2d)^{\frac{1}{2}}G(d) = \tfrac{1}{2}(2d-2)^{\frac{1}{2}}G(d).$$

These results for u and v may be restated in terms of the function

$$g(t) = 1 + \frac{1}{3}\,\frac{t}{4} + \frac{1\cdot 3}{2!\,5}\left(\frac{t}{4}\right)^2 + \frac{1\cdot 3\cdot 5}{3!\,7}\left(\frac{t}{4}\right)^3 + \cdots \qquad (16)$$

as follows:

$$u = \tfrac{1}{2}\pi - \sqrt{\frac{t}{2}}\,g(t), \qquad v = \sqrt{\frac{t}{2}}\,g(-t). \qquad (17)$$

The expressions (17) are obtained by setting $t = 1+c$ in the first and $t = d-1$ in the second.

In the real domain the circular and hyperbolic functions must be classed separately, since there are no real transformations interchanging the functions of these two classes. In the complex domain the functions must be classed together, since, as has already been observed, there exist complex transformations that interchange them. The same is true of their inverses; accordingly, the present tables can serve as tables of all inverse circular and hyperbolic functions in the complex domain. Use has already been made here of the very simple relations connecting inverse circular and hyperbolic sines and cosines. These relations may be shown in the form:

$$\text{arc cos } z = \tfrac{1}{2}\pi - \text{arc sin } z,$$

$$\text{arc sinh } z = -i \text{ arc sin } iz,$$

$$\text{arc cosh } z = -i \text{ arc cos } z = -i(\tfrac{1}{2}\pi - \text{arc sin } z).$$

Each of these equations is true whether it is interpreted as an equation relating principal values, or as an equation bearing on non-principal values. For example, the first equation may be taken as stating that the principal value of the arc cosine is obtained by subtracting from $\frac{1}{2}\pi$ the principal value of the arc sine; or, it may be taken as stating that if w is a number whose sine is z then $\frac{1}{2}\pi - w$ is a number whose cosine is

z, and that, moreover, as w runs through all values for which $\sin w$ is z then $\frac{1}{2}\pi - w$ runs through all values whose cosine is z.

The present tables can be used to evaluate arc tan z, but this evaluation requires a little care. One may write

$$\arctan z = \arcsin \frac{z}{\sqrt{1+z^2}} = \frac{1}{2} \arcsin \frac{2z}{1+z^2},$$

and thus express the arc tangent formally in terms of the arc sine in two different ways. The validity of each of these expressions may be verified purely formally by differentiation; in all three cases the derivative is $1/(1+z^2)$, and since all three functions have the same value at the origin, and have the same derivative, presumably the functions are identical. This gives a choice of two expressions for obtaining the arc tangent, and normally the second ought to be preferred, since it involves no square root extraction. Unfortunately, the second is also likely to give a value that is not the principal value of the arc tangent, despite the fact that the principal value of arc sin $2z/(1+z^2)$ is used to obtain it. In order to understand the correct use of the given formulas it is necessary to study the function arc tan z, and this will be done here by inverting the mapping $z = \tan w$.

This mapping is shown in Fig. 4. The period strip $-\frac{1}{2}\pi < w \leq \frac{1}{2}\pi$ is mapped into the entire z-plane with preservation of quadrants. If each plane is considered to have a point at infinity then the mapping is one-to-one with a single exception: the point w_∞ is mapped into the two points $z = \pm i$. Notice that the point $w = \frac{1}{2}\pi$ corresponds to the point z_∞. The kth period strip $k\pi-\frac{1}{2}\pi < w \leq k\pi+\frac{1}{2}\pi$ is likewise mapped into the entire z-plane, because of the periodicity

$$\tan (k\pi+w) = \tan w.$$

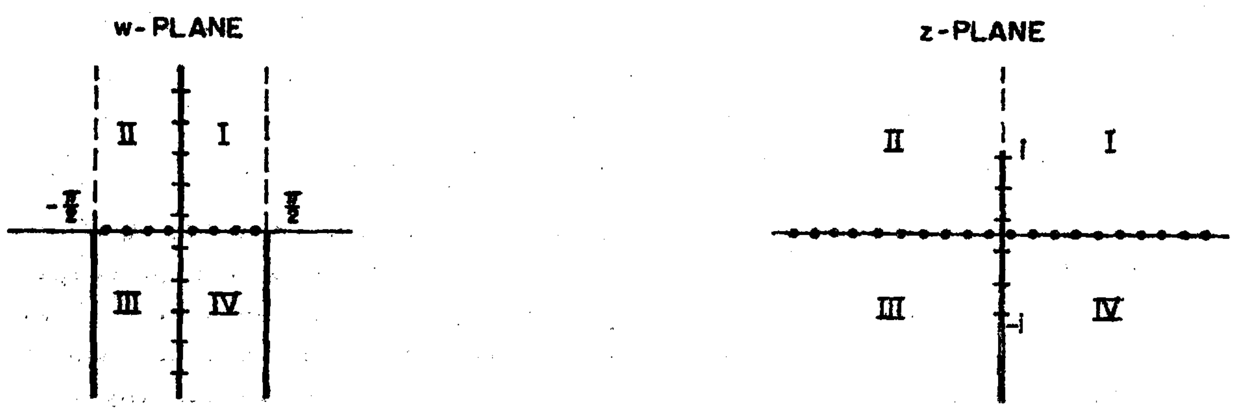

Fig. 4

The distribution of quarter-period-strips and quadrants is shown in Fig. 5.

The inverse function, w = arc tan z, may now be defined by inverting the mapping. It is an infinitely-multiple-valued function whose principal value may be defined as that value which lies in the principal period

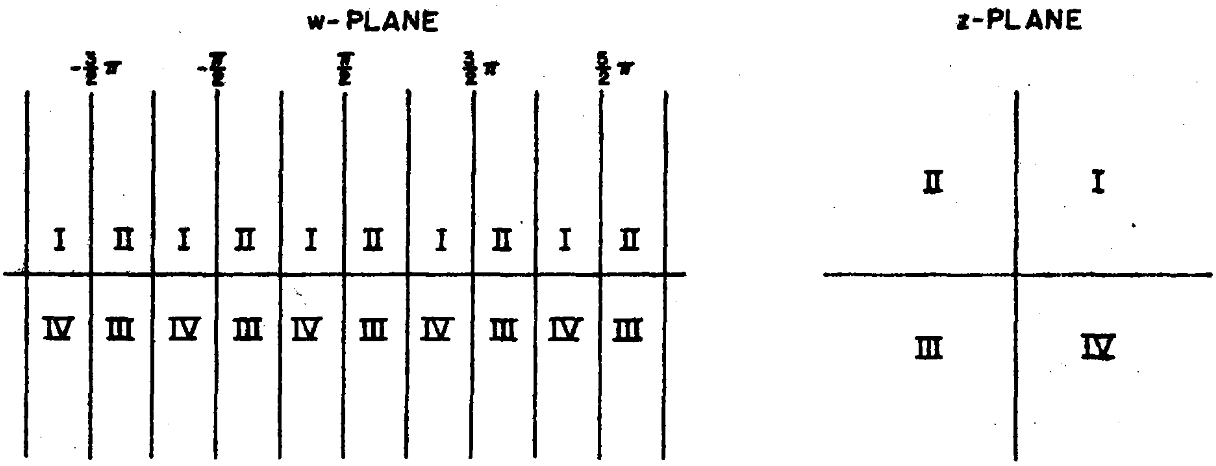

Fig. 5

strip $-\frac{1}{2}\pi < w \leq \frac{1}{2}\pi$. Moreover, if w = arc tan z, then as k runs through all rational integers the number $w+k\pi$ runs through all possible values of arc tan z. Note also the reflection properties, which state that if

$$u+vi = \text{arc tan } (x+yi),$$

then

$$u-vi = \text{arc tan } (x-yi),$$

$$-u+vi = \text{arc tan } (-x+yi),$$

$$-u-vi = \text{arc tan } (-x-yi).$$

It is seen that the problem of determining arc tan z may be referred to the case in which z is in the first quadrant.

If z is a number in the first quadrant, then so is $\zeta = z/\sqrt{1+z^2}$, and arc tan z may be obtained by use of the present tables once the number ζ has been calculated. The numerical labor is much simpler if instead one uses the formula

$$\text{arc tan } z = \tfrac{1}{2} \text{ arc sin } \xi,$$

with $\xi = 2z/(1+z^2)$. However, it may occur that ξ is in the fourth quadrant, and the required arc tan z must be a first quadrant number. To avoid this difficulty it is necessary to select from among the complex numbers arc sin $\xi+2k\pi$, $(2k+1)\pi$ - arc sin ξ, one that lies in the first quadrant. In other words, if z is in the first quadrant, arc tan z is

$$\left.\begin{matrix}\tfrac{1}{2}\text{ arc sin }\xi \\ \tfrac{1}{2}(\pi - \text{arc sin }\xi)\end{matrix}\right\}\text{ if }\xi\text{ is in the }\left\{\begin{matrix}\text{first} \\ \text{fourth}\end{matrix}\right\}\text{ quadrant.}$$

In this connection see Examples 6 and 7 at the end of Section III of this Introduction.

The determination of all other inverse trigonometric and hyperbolic functions in the complex domain may be referred to those here discussed, and hence to the present tables of the inverse sine. Thus

$$\text{arc tanh } z = -i \text{ arc tan } iz,$$

$$\text{arc sec } z = \text{arc cos } 1/z, \qquad \text{arc sech } z = \text{arc cosh } 1/z,$$

$$\text{arc csc } z = \text{arc sin } 1/z, \qquad \text{arc csch } z = \text{arc sinh } 1/z,$$

$$\text{arc cot } z = \text{arc tan } 1/z, \qquad \text{arc coth } z = \text{arc tanh } 1/z.$$

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INTRODUCTION

II

COMPOSITION OF THE TABLES

The discussion of the preceding section provides a basis for planning the arrangement and computation of a table of the function

$$f(z) = \text{arc sin } z.$$

It is sufficient to tabulate $f(z)$ in the first quadrant; however, in a finite tabulation only a limited portion of the quadrant can be tabulated. Moreover, it is not possible to give a fully interpolable tabulation of $f(z)$ in the vicinity of the singularity $z = 1$. In fact, along any line emanating from the branch point interpolation improves as the distance increases, so it cannot be expected that a single mesh on z will suffice for an efficient tabulation of the function. Accordingly, the present tables were planned as a set of tables on successively coarser meshes, covering regions of the plane at successively larger distances from the branch point.

Each table covers a region consisting of one, two, or three rectangles. To begin with, the region for the first table is a small rectangle containing the branch point $z = 1$. To form the region corresponding to the second table, one begins with a larger rectangle containing the branch point. The first rectangle is entirely contained in the second; by displacing it vertically until the overlap vanishes one arrives at a region which may be described as consisting of three rectangles, and this tri-rectangle is taken to be the domain of tabulation of the second table. The successive regions are formed in a similar manner. For the composition of the several tables see Figs. 6 and 7. In the execution of this plan slight modifications were made for reasons of typographical convenience.

In each table the tabular mesh is the same in both directions: $\Delta x = \Delta y = \delta$. Each table is composed of subtables. In a subtable x is fixed and y varies from y_0 through $y_0+70\Delta y$. In a large majority of cases y_0 is zero, but there are some cases where y_0 is different from zero, corresponding to the places where the nested rectangle was displaced to eliminate redundant tabulation.

Each subtable contains the real and imaginary parts of the function $f(z)$, together with $\delta f'(z)$, $\frac{1}{2}\delta^2 f''(z)$. Six decimal place values are given. All numbers were rounded off before printing. Each page of the book contains two complete subtables.

For computation, the quadrant was divided into complementary regions S and S'. S was the rectangle $y \leq 0.6$, $x \leq 1.35$. In S, u and v were calculated by use of the direct formulas (9).* In a form amenable to the Mark IV Calculator the latter read

$$u = \text{arc tan}\sqrt{\frac{1-c}{1+c}}$$

if c is positive,

$$u = \tfrac{1}{2}\pi - \text{arc tan}\sqrt{\frac{1+c}{1-c}}$$

if c is negative;

$$v = \tfrac{1}{2}\log_e\left(d + \sqrt{d^2-1}\right).$$

The variables c and d are functions of x and y defined by the formulas (7) and (8). The principal check of u and v was by the inverse functions

$$x = \cos\left(\tfrac{1}{2}\pi-u\right) \times \tfrac{1}{2}\left(e^{v} + \frac{1}{e^{v}}\right),$$

$$y = \cos u \times \tfrac{1}{2}\left(e^{v} - \frac{1}{e^{v}}\right),$$

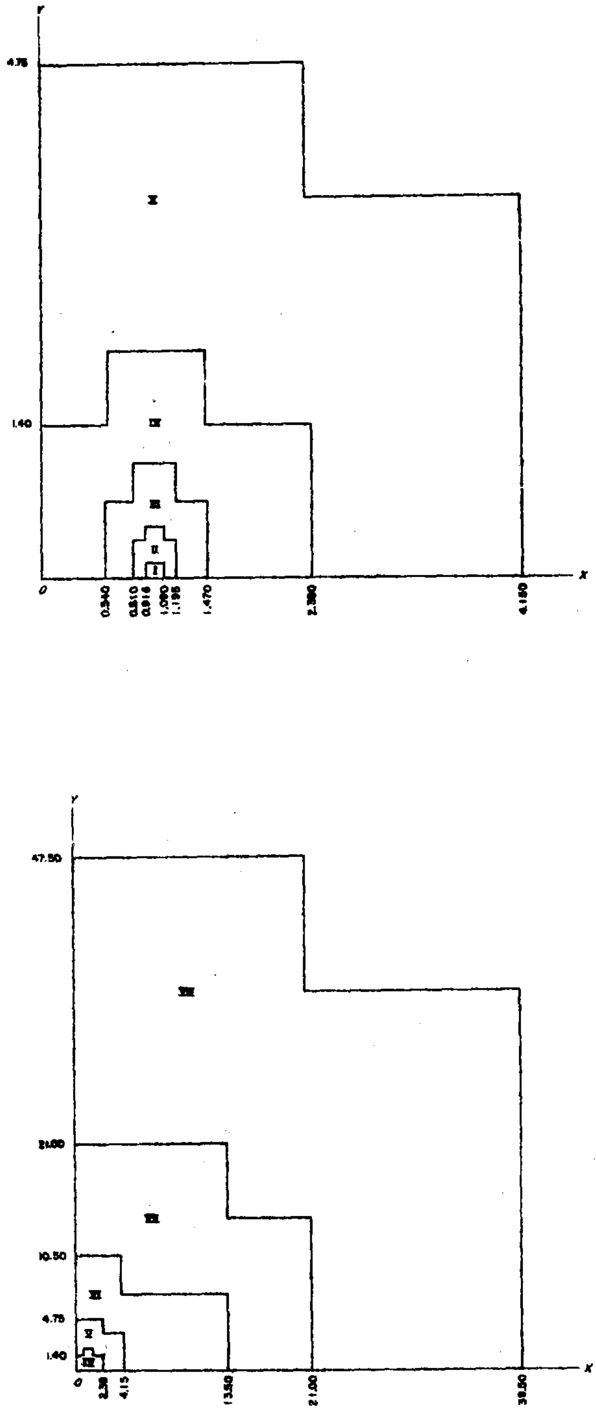

Fig. 6

which were applied with a tolerance of 10^{-12}. In this range the check could be expected to detect errors in u or v exceeding 10^{-9}.

* All equation numbers refer to equations in Section I.