Graduate Texts in Mathematics

Measure Theory

测度论

Paul R. Halmos

Springer-Verlag
世界图书出版公司

Paul R. Halmos

Measure Theory

Springer-Verlag

世界图书出版公司

北京·广州·上海·西安

AMS Subject Classifications (1970)
Primary: 28 - 02, 28A10, 28A15, 28A20, 28A25, 28A30, 28A35, 28A40, 28A60, 28A65, 28A70
Secondary: 60A05, 60Bxx

Library of Congress Cataloging in Publication Data

Halmos, Paul Richard, 1914-
Measure theory.

(Graduate texts in mathematics, 18)
Reprint of the ed. published by Van Nostrand, New York, in series: The University series in higher mathematics.
Bibliography: p.
1. Measure theory. I. Title. II. Series.
[QA312.H26 1974] 515'.42 74-10690

This reprint has been authorized by Springer-Verlag (Berlin/Heidelberg/New York) for sale in the People's Republic of China only and not for export therefrom.
Reprinted in China by Beijing World Publishing Corporation, 1998

ISBN 0-387-90088-8 Springer-Verlag New York Heidelberg Berlin
ISBN 3-540-90088-8 Springer-Verlag Berlin Heidelberg New York

PREFACE

My main purpose in this book is to present a unified treatment of that part of measure theory which in recent years has shown itself to be most useful for its applications in modern analysis. If I have accomplished my purpose, then the book should be found usable both as a text for students and as a source of reference for the more advanced mathematician.

I have tried to keep to a minimum the amount of new and unusual terminology and notation. In the few places where my nomenclature differs from that in the existing literature of measure theory, I was motivated by an attempt to harmonize with the usage of other parts of mathematics. There are, for instance, sound algebraic reasons for using the terms "lattice" and "ring" for certain classes of sets—reasons which are more cogent than the similarities that caused Hausdorff to use "ring" and "field."

The only necessary prerequisite for an intelligent reading of the first seven chapters of this book is what is known in the United States as undergraduate algebra and analysis. For the convenience of the reader, § 0 is devoted to a detailed listing of exactly what knowledge is assumed in the various chapters. The beginner should be warned that some of the words and symbols in the latter part of § 0 are defined only later, in the first seven chapters of the text, and that, accordingly, he should not be discouraged if, on first reading of § 0, he finds that he does not have the prerequisites for reading the prerequisites.

At the end of almost every section there is a set of exercises which appear sometimes as questions but more usually as assertions that the reader is invited to prove. These exercises should be viewed as corollaries to and sidelights on the results more formally expounded. They constitute an integral part of the book; among them appear not only most of the examples and counter examples necessary for understanding the theory, but also definitions of new concepts and, occasionally, entire theories that not long ago were still subjects of research.

It might appear inconsistent that, in the text, many elementary notions are treated in great detail, while, in the exercises, some quite refined and profound matters (topological spaces, transfinite numbers, Banach spaces, etc.) are assumed to be known. The material is arranged, however, so that when a beginning student comes to an exercise which uses terms not defined in this book he may simply omit it without loss of continuity. The more advanced reader, on the other hand, might be pleased at the interplay between measure theory and other parts of mathematics which it is the purpose of such exercises to exhibit.

The symbol ▮ is used throughout the entire book in place of such phrases as "Q.E.D." or "This completes the proof of the theorem" to signal the end of a proof.

At the end of the book there is a short list of references and a bibliography. I make no claims of completeness for these lists. Their purpose is sometimes to mention background reading, rarely (in cases where the history of the subject is not too well known) to give credit for original discoveries, and most often to indicate directions for further study.

A symbol such as $u.v$, where u is an integer and v is an integer or a letter of the alphabet, refers to the (unique) theorem, formula, or exercise in section u which bears the label v.

ACKNOWLEDGMENTS

Most of the work on this book was done in the academic year 1947–1948 while I was a fellow of the John Simon Guggenheim Memorial Foundation, in residence at the Institute for Advanced Study, on leave from the University of Chicago.

I am very much indebted to D. Blackwell, J. L. Doob, W. H. Gottschalk, L. Nachbin, B. J. Pettis, and, especially, to J. C. Oxtoby for their critical reading of the manuscript and their many valuable suggestions for improvements.

The result of 3.13 was communicated to me by E. Bishop. The condition in 31.10 was suggested by J. C. Oxtoby. The example 52.10 was discovered by J. Dieudonné.

P. R. H.

书　　名：Measure Theory
作　　者：Paul R.Halmos
中 译 名：测度论
出 版 者：世界图书出版公司北京公司
印 刷 者：北京中西印刷厂
发　　行：世界图书出版公司北京公司（北京朝阳门内大街 137 号　100010）
开　　本：大 32 开 850 × 1168　印　张：10
版　　次：1998 年 8 月第 1 版　1998 年 8 月第 1 次印刷
书　　号：7-5062-0048-1
定　　价：45.00 元

CONTENTS

§ 0. PREREQUISITES

The only prerequisite for reading and understanding the first seven chapters of this book is a knowledge of elementary algebra and analysis. Specifically it is assumed that the reader is familiar with the concepts and results listed in (1)–(7) below.

(1) Mathematical induction, commutativity and associativity of algebraic operations, linear combinations, equivalence relations and decompositions into equivalence classes.

(2) Countable sets; the union of countably many countable sets is countable.

(3) Real numbers, elementary metric and topological properties of the real line (e.g. the rational numbers are dense, every open set is a countable union of disjoint open intervals), the Heine–Borel theorem.

(4) The general concept of a function and, in particular, of a sequence (i.e. a function whose domain of definition is the set of positive integers); sums, products, constant multiples, and absolute values of functions.

(5) Least upper and greatest lower bounds (called suprema and infima) of sets of real numbers and real valued functions; limits, superior limits, and inferior limits of sequences of real numbers and real valued functions.

(6) The symbols $+\infty$ and $-\infty$, and the following algebraic relations among them and real numbers x:

$$(\pm\infty) + (\pm\infty) = x + (\pm\infty) = (\pm\infty) + x = \pm\infty;$$

$$x(\pm\infty) = (\pm\infty)x = \begin{cases} \pm\infty & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ \mp\infty & \text{if } x < 0; \end{cases}$$

$$(\pm\infty)(\pm\infty) = +\infty,$$

$$(\pm\infty)(\mp\infty) = -\infty;$$

$$x/(\pm\infty) = 0;$$

$$-\infty < x < +\infty.$$

The phrase **extended real number** refers to a real number or one of the symbols $\pm\infty$.

(7) If x and y are real numbers,

$$x \cup y = \max\{x,y\} = \tfrac{1}{2}(x + y + |x - y|),$$

$$x \cap y = \min\{x,y\} = \tfrac{1}{2}(x + y - |x - y|).$$

Similarly, if f and g are real valued functions, then $f \cup g$ and $f \cap g$ are the functions defined by

$$(f \cup g)(x) = f(x) \cup g(x) \quad \text{and} \quad (f \cap g)(x) = f(x) \cap g(x),$$

respectively. The supremum and infimum of a sequence $\{x_n\}$ of real numbers are denoted by

$$\bigcup_{n=1}^{\infty} x_n \quad \text{and} \quad \bigcap_{n=1}^{\infty} x_n,$$

respectively. In this notation

$$\limsup_n x_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} x_m$$

and

$$\liminf_n x_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} x_m.$$

In Chapter VIII the concept of metric space is used, together with such related concepts as completeness and separability for metric spaces, and uniform continuity of functions on metric spaces. In Chapter VIII use is made also of such slightly more sophisticated concepts of real analysis as one-sided continuity.

In the last section of Chapter IX, Tychonoff's theorem on the compactness of product spaces is needed (for countably many factors each of which is an interval).

In general, each chapter makes free use of all preceding chapters; the only major exception to this is that Chapter IX is not needed for the last three chapters.

In Chapters X, XI, and XII systematic use is made of many of the concepts and results of point set topology and the elements of topological group theory. We append below a list of all the relevant definitions and theorems. The purpose of this list is not to serve as a text on topology, but (a) to tell the expert exactly which forms of the relevant concepts and results we need, (b) to tell the beginner with exactly which concepts and results he should familiarize himself before studying the last three chapters, (c) to put on record certain, not universally used, terminological conventions, and (d) to serve as an easily available reference for things which the reader may wish to recall.

Topological Spaces

A **topological space** is a set X and a class of subsets of X, called the **open sets** of X, such that the class contains 0 and X and is closed under the formation of finite intersections and arbitrary (i.e. not necessarily finite or countable) unions. A subset E of X is called a G_δ if there exists a sequence $\{U_n\}$ of open sets such that $E = \bigcap_{n=1}^{\infty} U_n$. The class of all G_δ's is closed under the formation of finite unions and countable intersections. The topological space X is **discrete** if every subset of X is open, or, equivalently, if every one-point subset of X is open. A set E is **closed** if $X - E$ is open. The class of closed sets contains 0 and X and is closed under the formation of finite unions and arbitrary intersections. The **interior,** E^0, of a subset E of X is the greatest open set contained in E; the **closure,** $\bar{E}$, of E is the least closed set containing E. Interiors are open sets and closures are closed sets; if E is open, then $E^0 = E$, and, if E is closed, then $\bar{E} = E$. The closure of a set E is the set of all points x such that, for every open set U containing x, $E \cap U \neq 0$. A set E is **dense** in X if $\bar{E} = X$. A subset Y of a topological space becomes a topological space (a **subspace** of X) in the **relative topology** if exactly those subsets of Y are called open which may be obtained by intersecting an open subset of X with Y. A **neighborhood** of a point x in X [or of a subset E of X] is an open set containing x [or an open set containing E]. A **base** is a class **B** of open sets such that, for every x in X and every neighborhood U of x, there exists a set B in **B** such that $x \in B \subset U$. The **topology of the real line** is determined by the requirement that the class of all open intervals be a base. A **subbase** is a class of sets, the class of all finite intersections of which is a base. A space X is **separable** if it has a countable base. A subspace of a separable space is separable.

An **open covering** of a subset E of a topological space X is a class $\mathbf{K}$ of open sets such that $E \subset \bigcup \mathbf{K}$. If X is separable and $\mathbf{K}$ is an open covering of a subset E of X, then there exists a countable subclass $\{K_1, K_2, \cdots\}$ of $\mathbf{K}$ which is an open covering of E. A set E in X is **compact** if, for every open covering $\mathbf{K}$ of E, there exists a finite subclass $\{K_1, \cdots, K_n\}$ of $\mathbf{K}$ which is an open covering of E. A class $\mathbf{K}$ of sets has the **finite intersection property** if every finite subclass of $\mathbf{K}$ has a non empty intersection. A space X is compact if and only if every class of closed sets with the finite intersection property has a non empty intersection. A set E in a space X is **σ–compact** if there exists a sequence $\{C_n\}$ of compact sets such that $E = \bigcup_{n=1}^{\infty} C_n$. A space X is **locally compact** if every point of X has a neighborhood whose closure is compact. A subset E of a locally compact space is **bounded** if there exists a compact set C such that $E \subset C$. The class of all bounded open sets in a locally compact space is a base. A closed subset of a bounded set is compact. A subset E of a locally compact space is **σ–bounded** if there exists a sequence $\{C_n\}$ of compact sets such that $E \subset \bigcup_{n=1}^{\infty} C_n$. To any locally compact but not compact topological space X there corresponds a compact space X^* containing X and exactly one additional point x^*; X^* is called the **one-point compactification** of X by x^*. The open sets of X^* are the open subsets of X and the complements (in X^*) of the closed compact subsets of X.

If $\{X_i\colon i \in I\}$ is a class of topological spaces, their **Cartesian product** is the set $X = \times\{X_i\colon i \in I\}$ of all functions x defined on I and such that, for each i in I, $x(i) \in X_i$. For a fixed i_0 in I, let E_{i_0} be an open subset of X_{i_0}, and, for $i \neq i_0$, write $E_i = X_i$; the class of open sets in X is determined by the requirement that the class of all sets of the form $\times\{E_i\colon i \in I\}$ be a subbase. If the function ξ_i on X is defined by $\xi_i(x) = x(i)$, then ξ_i is continuous. The Cartesian product of any class of compact spaces is compact.

A topological space is a **Hausdorff space** if every pair of distinct points have disjoint neighborhoods. Two disjoint compact sets in a Hausdorff space have disjoint neighborhoods. A compact subset of a Hausdorff space is closed. If a locally compact space is a Hausdorff space or a separable space, then so is its one-point compactification. A real valued continuous function on a compact set is bounded.

For any topological space X we denote by $\mathfrak{F}$ (or $\mathfrak{F}(X)$) the class of all real valued continuous functions f such that $0 \leqq f(x) \leqq 1$ for all x in X. A Hausdorff space is **completely regular** if, for every point y in X and every closed set F not containing y, there is a function f in $\mathfrak{F}$ such that $f(y) = 0$ and, for x in F, $f(x) = 1$. A locally compact Hausdorff space is completely regular.

A **metric space** is a set X and a real valued function d (called **distance**) on $X \times X$, such that $d(x,y) \geqq 0$, $d(x,y) = 0$ if and only if $x = y$, $d(x,y) = d(y,x)$, and $d(x,y) \leqq d(x,z) + d(z,y)$. If E and F are non empty subsets of a metric space X, the distance between them is defined to be the number $d(E,F) = \inf \{d(x,y)\colon x \,\varepsilon\, E, y \,\varepsilon\, F\}$. If $F = \{x_0\}$ is a one-point set, we write $d(E,x_0)$ in place of $d(E,\{x_0\})$. A **sphere** (with **center** x_0 and **radius** r_0) is a subset E of a metric space X such that, for some point x_0 and some positive number r_0, $E = \{x\colon d(x_0,x) < r_0\}$. The **topology of a metric space** is determined by the requirement that the class of all spheres be a base. A metric space is completely regular. A closed set in a metric space is a G_δ. A metric space is separable if and only if it contains a countable dense set. If E is a subset of a metric space and $f(x) = d(E,x)$, then f is a continuous function and $\bar{E} = \{x\colon f(x) = 0\}$. If X is the real line, or the Cartesian product of a finite number of real lines, then X is a locally compact separable Hausdorff space; it is even a metric space if for $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ the distance $d(x,y)$ is defined to be $(\sum_{i=1}^{n} (x_i - y_i)^2)^{1/2}$. A closed interval in the real line is a compact set.

A transformation T from a topological space X into a topological space Y is **continuous** if the inverse image of every open set is open, or, equivalently, if the inverse image of every closed set is closed. The transformation T is **open** if the image of every open set is open. If $\mathbf{B}$ is a subbase in Y, then a necessary and sufficient condition that T be continuous is that $T^{-1}(B)$ be open for every B in $\mathbf{B}$. If a continuous transformation T maps X onto Y, and if X is compact, then Y is compact. A **homeomorphism** is a one *to one, continuous transformation of X onto Y whose inverse is* also continuous.

The sum of a uniformly convergent series of real valued, continuous functions is continuous. If f and g are real valued continuous functions, then $f \cup g$ and $f \cap g$ are continuous.

Topological Groups

A **group** is a non empty set X of elements for which an associative multiplication is defined so that, for any two elements a and b of X, the equations $ax = b$ and $ya = b$ are solvable. In every group X there is a unique **identity** element e, characterized by the fact that $ex = xe = x$ for every x in X. Each element x of X has a unique **inverse,** x^{-1}, characterized by the fact that $xx^{-1} = x^{-1}x = e$. A non empty subset Y of X is a **subgroup** if $x^{-1}y \varepsilon Y$ whenever x and y are in Y. If E is any subset of a group X, E^{-1} is the set of all elements of the form x^{-1}, where $x \varepsilon E$; if E and F are any two subsets of X, EF is the set of all elements of the form xy, where $x \varepsilon E$ and $y \varepsilon F$. A non empty subset Y of X is a subgroup if and only if $Y^{-1}Y \subset Y$. If $x \varepsilon X$, it is customary to write xE and Ex in place of $\{x\}E$ and $E\{x\}$ respectively; the set xE [or Ex] is called a **left translation** [or **right translation**] of E. If Y is a subgroup of X, the sets xY and Yx are called (left and right) **cosets** of Y. A subgroup Y of X is **invariant** if $xY = Yx$ for every x in X. If the product of two cosets Y_1 and Y_2 of an invariant subgroup Y is defined to be Y_1Y_2, then, with respect to this notion of multiplication, the class of all cosets is a group $\hat{X}$, called the **quotient group** of X modulo Y and denoted by X/Y. The identity element $\hat{e}$ of $\hat{X}$ is Y. If Y is an invariant subgroup of X, and if for every x in X, $\pi(x)$ is the coset of Y which contains x, then the transformation π is called the **projection** from X onto $\hat{X}$. A **homomorphism** is a transformation T from a group X into a group Y such that $T(xy) = T(x)T(y)$ for every two elements x and y of X. The projection from a group X onto a quotient group $\hat{X}$ is a homomorphism.

A **topological group** is a group X which is a Hausdorff space such that the transformation (from $X \times X$ onto X) which sends

(x,y) into $x^{-1}y$ is continuous. A class **N** of open sets containing e in a topological group is a **base at** e if (a) for every x different from e there exists a set U in **N** such that $x \varepsilon' U$, (b) for any two sets U and V in **N** there exists a set W in **N** such that $W \subset U \cap V$, (c) for any set U in **N** there exists a set V in **N** such that $V^{-1}V \subset U$, (d) for any set U in **N** and any element x in X, there exists a set V in **N** such that $V \subset xUx^{-1}$, and (e) for any set U in **N** and any element x in U there exists a set V in **N** such that $Vx \subset U$. The class of all neighborhoods of e is a base at e; conversely if, in any group X, **N** is a class of sets satisfying the conditions described above, and if the class of all translations of sets of **N** is taken for a base, then, with respect to the topology so defined, X becomes a topological group. A neighborhood V of e is **symmetric** if $V = V^{-1}$; the class of all symmetric neighborhoods of e is a base at e. If **N** is a base at e and if F is any closed set in X, then $F = \bigcap \{UF: U \varepsilon \mathbf{N}\}$.

The closure of a subgroup [or of an invariant subgroup] of a topological group X is a subgroup [or an invariant subgroup] of X. If Y is a closed invariant subgroup of X, and if a subset of $\mathfrak{X} = X/Y$ is called open if and only if its inverse image (under the projection π) is open in X, then $\mathfrak{X}$ is a topological group and the transformation π from X onto $\mathfrak{X}$ is open and continuous.

If C is a compact set and U is an open set in a topological group X, and if $C \subset U$, then there exists a neighborhood V of e such that $VCV \subset U$. If C and D are two disjoint compact sets, then there exists a neighborhood U of e such that UCU and UDU are disjoint. If C and D are any two compact sets, then C^{-1} and CD are also compact.

A subset E of a topological group X is **bounded** if, for every neighborhood U of e, there exists a finite set $\{x_1, \cdots, x_n\}$ (which, in case $E \neq 0$, may be assumed to be a subset of E) such that $E \subset \bigcup_{i=1}^{n} x_iU$; if X is locally compact, then this definition of boundedness agrees with the one applicable in any locally compact space (i.e. the one which requires that the closure of E be compact). If a continuous, real valued function f on X is such that the set $N(f) = \{x: f(x) \neq 0\}$ is bounded, then f is **uniformly continuous** in the sense that to every positive number ϵ there