NETWORK THEORY

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PREFACE

This book is intended for an advanced undergraduate or beginning graduate course in network theory. At the undergraduate level, for the student with reasonable facility in mathematics, it can be used following a basic course in network or circuit analysis. At the graduate level it should be used during the first semester of the student's graduate program.

The University of New Hampshire, as is typical of many small colleges and universities, has a modest master's degree program in electrical engineering. The students enrolled in this program, often on an extension or part-time basis, generally have a different purpose or background from students at larger institutions with sizeable resident programs offering the doctorate. Also they may have been out of college for several years while working in industry. In particular their backgrounds in networks and associated mathematics are often somewhat weak in one way or another. There are at least two ways by which this deficiency may be corrected:

1. A remedial course can be offered which covers the prerequisite undergraduate network theory and mathematics. One disadvantage of this

seems to be that the incoming graduate students are not homogeneous in their deficiencies. One student may be able to handle Laplace transforms well but know little about matrices; a second student's background may be just the reverse. Thus a remedial course, in trying to serve everyone, may actually not serve anyone properly. A second disadvantage is that the student's graduate program is set back a semester. This discourages many industrial students from pursuing a master's degree.

2. A first graduate course can be offered that reviews undergraduate network theory but does it from a different point of view, more rigorously, and in greater depth. Ideally, the deficient student (assuming his background is not too weak) erases his deficiency and at the same time develops new methods and tools of mathematics and network analysis, whereas the already reasonably prepared student deepens his understanding of network theory, polishes his math, develops the habit of reading the literature, and acquires a thesis topic.

This second approach of reviewing in depth and extending has been used at the University of New Hampshire for several years. Incoming electrical-engineering graduate students take two required courses in their first semester, one in field theory and the other in network theory. These courses seek to bring the student to a reasonable level of proficiency in fields and networks while at the same time sharpening the mathematical tools he needs for subsequent graduate courses. Thus partial differential equations, boundary-value problems, Bessel and Legendre functions, and conformal transformations are included in the fields course, whereas matrices and determinants, systems of equations, linear-graph theory, Laplace and Fourier transforms, and complex and state variables are stressed in the networks course.

This book has evolved from notes for this first graduate course in network theory and has been classroom-tested for three years. Roughly the first half of the book concentrates on the steady-state analysis of linear networks from a matrix and topological point of view, and the last half of the book considers the free and forced behavior of linear networks, using Laplace and Fourier transforms and state and complex variables. Chapter 8, on Natural Frequencies, serves as a bridge between these two areas. Sufficient depth is provided in each section of the book to ensure a student being able to read related material in the literature intelligently. At the same time, sufficient review is included to ensure that the deficient student is not overwhelmed. Chapter problems are divided into three categories in keeping with this philosophy: drill (a graduate student does need some), theory and proofs, and application.

The book is too long to be covered completely in a single semester. However, there is adequate time in a single semester to cover either one of the two major topics in detail while treating the second lightly. Alternatively, both major topics can be covered with some thoroughness in one semester by omitting several special and advanced topics. In terms of chapter and section coverage, the three possibilities are as follows:

Accent on steady-state analysis, topology, and matrices Chaps. 1 to 7, Secs. 8.2, 8.5, 8.9 to 8.11, 9.2, 9.3, 10.5, 10.6, 11.4, and 11.7

Accent on free and forced response, Laplace and Fourier transforms, and state and complex variables Chaps. 1 and 2, Secs. 3.1 to 3.5, 7.1 to 7.4, Chap. 8 (omit 8.5), Chaps. 9 and 10, Chap. 11 (omit 11.4), Chap. 12

Balanced emphasis on each topic Chap. 1 (omit 1.11 and 1.14), Chap. 2 (omit 2.9 and 2.10), Chap. 3 (omit 3.6 to 3.10), Chap. 4 (omit 4.9 to 4.13), Chap. 5 (omit 5.7 to 5.11), Chap. 7 (omit 7.5 and 7.6), Chap. 8 (omit 8.9 to 8.11), Chap. 9, Chap. 10 (omit 10.2, 10.5 and 10.7), Chap. 11 (omit 11.4 and 11.6 to 11.8), Chap. 12 (omit 12.9 to 12.16)

Of course, the book can be used for a full-year course, topology and matrices being accented during the first semester and transforms and state and complex variables during the second. At the graduate level, the instructor will probably have time each semester to supplement the book with topics of his own choosing and interest. At the undergraduate level, the book itself should suffice.

It is a pleasure to express thanks to the two people who introduced me to the fascinating business of networks during my Ph.D. days, Dr. Dov Hazony and Dr. James D. Schoeffler, of Case Institute of Technology. I am also indebted to Prof. Donald W. Melvin, of the University of New Hampshire, who classroom-tested and commented most constructively on various sections of the book. To Mrs. Barbara Keating, who typed the early chapters, and to Mrs. Mary Ann Stickney, who came along at just the right time and devoted herself to the typing of the later chapters and the preparation of the final manuscript, I acknowledge a sincere debt of gratitude. Finally, to my graduate students, who suffered through uncorrected ditto copy, my appreciation for suggestions and corrections.

J. B. MURDOCH

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1 MATRICES AND DETERMINANTS

1.1 INTRODUCTION

A linear system is one that can be characterized by a set of linear differential or algebraic equations. For a linear electric network, these equations are the Kirchhoff law equations resulting from a loop or node analysis of the network. The reader is assumed to have a working knowledge of the formulation and solution of such equations. Specifically, it is assumed that he can write Kirchhoff's voltage law (KVL) equations around the various loops of a network and Kirchhoff's current law (KCL) equations at the various nodes and, presuming the right number of equations has been written, that he can solve for the unknown currents or voltages using determinants and Cramer's rule.

Simple networks are easily analyzed in the manner just described. Questions about the number of necessary equations seldom arise. However, as the complexity of the network increases, a more systematic approach is needed. It is the purpose of the first several chapters to develop such an approach based on linear graph theory. Matrices and determinants are particularly useful mathematical tools in this development. We therefore devote the first chapter to their study.

The reader is assumed to have some familiarity with matrices and determinants. Our purpose here is to review their properties, develop a facility in and understanding of their use, and emphasize those features and special forms which are of particular importance in the study of linear graph theory.

1.2 DEFINITION OF A MATRIX

Consider a set of m linear algebraic equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= y_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= y_m \end{aligned} \tag{1.2.1}$$

We ask: How can these equations be written more concisely? To answer the question, we rewrite Eq. (1.2.1) in the *matrix* form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \equiv \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{bmatrix} \tag{1.2.2}$$

which can be shortened to

$$AX \equiv Y \tag{1.2.3}$$

Equations (1.2.2) and (1.2.3) are definitions. Each is a shorthand form of Eq. (1.2.1). Matrix A is an ordered array of coefficients which transforms a set of n variables in x into a set of m variables in y, as prescribed by Eq. (1.2.1). It has m rows and n columns and is said to be of *order* $m \times n$. Similarly X and Y are *column* matrices of order $n \times 1$ and $m \times 1$, respectively.

1.3 INDEX NOTATION

Thus far we have represented a matrix either by a single capital letter or by a bracketed array of single- or double-subscripted coefficients. A third, very useful representation is provided by index notation. Let

$$\begin{aligned} A &\equiv [a_{ij}] \\ X &\equiv [x_i] \\ Y &\equiv [y_i] \end{aligned} \tag{1.3.1}$$

In this notation, a_{ij} is the element in the ith row and jth column of A, and $[a_{ij}]$ is the matrix composed of these elements. Similarly x_i is the element in the ith row of the matrix X (there being only one column), and likewise for y_i.

The definitions of Eq. (1.3.1) can be substituted into Eq. (1.2.3) to obtain

$$[a_{ij}][x_i] = [y_i] \tag{1.3.2}$$

Equation (1.3.2) makes little sense in terms of the instructions provided by the indices. The index i is repeated on the left of Eq. (1.3.2) but not on the right; there is nothing in the notation to guide us in multiplying the two matrices on the left together.

To make Eq. (1.3.2) meaningful, let us write Eq. (1.2.1) as a summation using index notation

$$\sum_{k=1}^{n} a_{ik}x_k = y_i \qquad i = 1, 2, \ldots, m \tag{1.3.3}$$

For each value of i, one equation of Eq. (1.2.1) is developed as k proceeds from 1 through n. Putting Eq. (1.3.3) in matrix form yields the entire set of equations

$$\,^{m}_{i=1}\left[\sum_{k=1}^{n} a_{ik}x_k\right] = \,^{m}_{i=1}[y_i] \tag{1.3.4}$$

The brackets in Eq. (1.3.4) tell us to let i take on all its permitted values from 1 through m.

The repeated subscript k in Eqs. (1.3.3) and (1.3.4) is known as a *dummy index*, indicating a summation over the n columns of A and rows of X. Clearly the number of columns in the first matrix and rows in the second matrix must be the same for this matrix product to be defined. Matrices satisfying this requirement are said to be *conformable* in the order given.

There is no reason to retain the summation sign in Eq. (1.3.4) if we are willing to adopt the convention that a repeated index indicates a summation. (In the few cases where a repeated index does not indicate a summation we shall use a Greek letter for the repeated index.) Also, the ranges of i and k are customarily omitted [as they are in Eq. (1.2.3)]. With these changes, Eq. (1.3.4) becomes

$$[a_{ik}x_k] = [y_i] \tag{1.3.5}$$

Let us now return to our difficulties with Eq. (1.3.2) and change this equation to read

$$[a_{ik}][x_k] = [y_i] \tag{1.3.6}$$

The form in Eq. (1.3.6) looks much better. Formally, we have replaced the second (column) index of the first matrix and the first (row) index of the second matrix by a dummy summation index (in this case k).

Now comparing Eqs. (1.3.5) and (1.3.6), we obtain

$$[a_{ik}][x_k] = [a_{ik}x_k] \tag{1.3.7}$$

which shows, in compact index-notation form, exactly what is meant by the product of two matrices.

1.4 MULTIPLICATION

To develop further what is meant by matrix multiplication and show how several matrices can be multiplied together (*cascaded*), let B be an $n \times p$ matrix given by

$$B = [b_{ij}] \tag{1.4.1}$$

and let Z be a $p \times 1$ column matrix given by

$$Z = [z_i] \tag{1.4.2}$$

Now let the n variables in x in Eq. (1.2.1) be defined by n equations in the p variables in z by

$$[b_{il}][z_l] = [x_i] \qquad \text{or} \qquad BZ = X \tag{1.4.3}$$

where l is a dummy summation index like k in Eq. (1.3.2) but runs from 1 through p. Substituting Eq. (1.4.3), with i replaced by k, into Eq. (1.3.6) yields

$$[a_{ik}][b_{kl}][z_l] = [y_i] \tag{1.4.4}$$

which, according to Eq. (1.3.7), can be written as

$$[a_{ik}b_{kl}][z_l] = [y_i] \tag{1.4.5}$$

On the left in Eqs. (1.4.4) and (1.4.5) there are two dummy summation indices, namely, k and l, and there is one nonrepeated index i. The summation indices do not appear on the right; the nonrepeated index does. This is always the case.

Now define

$$[c_{il}] \equiv [a_{ik}b_{kl}] \tag{1.4.6}$$

which yields

$$[c_{il}][z_l] = [y_i] \tag{1.4.7}$$

From Eq. (1.4.6), we can formulate a general procedure for taking a matrix product AB. First A and B must be conformable. Here A is an $m \times n$ matrix and B is an $n \times p$ matrix. They are conformable, and their product is an $m \times p$ matrix. Second, each entry in the C matrix is formed by letting k take on all values from 1 through n, with i and l fixed, and summing the products. For example,

$$c_{12} = a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1n}b_{n2} \tag{1.4.8}$$

Example 1.1 Let the matrices A, B, Z, and Y in Eq. (1.4.4) be given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \qquad Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

We can write any of the y_i's by inspection using index notation and Eq. (1.4.4). For example, to write y_2, we note that $i = 2$ and k and l each take on the values 1 and 2. Thus

$$\begin{aligned} y_2 &= a_{2k}b_{kl}z_l \\ &= a_{21}b_{11}z_1 + a_{21}b_{12}z_2 + a_{22}b_{21}z_1 + a_{22}b_{22}z_2 \end{aligned}$$

Note that by using index notation, we did not have to perform the multiplication piecemeal, i.e., by first obtaining AB and then ABZ, or BZ and then ABZ. This can be an advantage when several matrices are to be multiplied together.

Example 1.2 The two networks shown in Fig. 1.1 are to be cascaded as indicated by the dashed lines. We wish to form the overall network matrix.

The voltage-current equations† for each network are

$$e_1 = ae_2 - bi_2 \qquad e_3 = a'e_4 - b'i_4$$
$$i_1 = ce_2 - di_2 \qquad i_3 = c'e_4 - d'i_4$$

When the networks are cascaded,

$$e_3 = e_2 \qquad \text{and} \qquad i_3 = -i_2$$

Hence the equations for the second network become

$$e_2 = a'e_4 - b'i_4$$
$$-i_2 = c'e_4 - d'i_4$$

† The equations are given in terms of the chain parameters of each network. Two-port network parameters are discussed in Chap. 3.

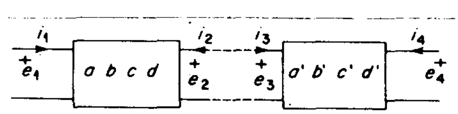

Fig. 1.1 Cascaded networks.