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Linear Operator Theory in Engineering and Science

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With 120 Figures

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Preface

The goal of this book is to present the basic facts of functional analysis in a form suitable for engineers, scientists, and applied mathematicians. Although the Definition–Theorem–Proof format of mathematics is used, careful attention is given to motivation of the material covered and many illustrative examples are presented.

The text can be used by students with various levels of preparation. However, the typical student is probably a first-year graduate student in engineering, one of the formal sciences, or mathematics. It is also possible to use this book as a text for a senior-level course. In order to facilitate students with varying backgrounds, a number of appendices covering useful mathematical topics have been included. Moreover, there has also been an attempt to make the pace in the beginning more gradual than that of later chapters.

The first five chapters are concerned with the "geometry" of normed linear spaces. The basic approach is to "disassemble" this geometric structure first, study the pieces, then reassemble and study the whole geometry. The pieces that result from this disassembly are set-theoretic, topological, and algebraic structures. Hence, Chapter 2 covers the appropriate set theory; Chapter 3 treats topological structure, in particular, metric spaces; and Chapter 4 handles algebraic structure, in particular, linear spaces. The reassembly takes place in Chapter 5 where normed linear spaces are studied. The main topic of this chapter is the geometry of Hilbert spaces.

The authors have found that the material covered in these first five chapters can be presented in a one-semester beginning graduate course. Indeed, the authors have done so a number of times in engineering and mathematics departments at a number of universities in the United States, Europe, and South America. Needless to say, the mode of presentation depends upon the audience. For certain audiences, motivation and examples are emphasized while proofs are only highlighted. For others, the converse is the case. An attempt has been made to make the book suitable for both modes of presentation. Moreover, there is material in the large collection of exercises appropriate for each type of audience.

Chapters 6 and 7 take the geometric structure developed in the first five chapters and apply it to the geometric analysis of linear operators. Chapter 6 covers the Spectral Theorem (the eigenvalue-eigenvector representation) for compact operators. Chapter 7 extends this material to certain discontinuous operators, in particular it treats those operators with compact resolvents. These two chapters also contain many illustrative examples.

Many chapters are divided into parts (Part A, Part B, and so forth). Part A contains basic introductory concepts. The subsequent parts of each chapter develop additional concepts and special topics. Thus, if a relatively quick introduction is desired, Part A can be covered first and material from the rest of the chapter can be added as needed.

For the person who is interested in getting to the spectral theory of linear operators as soon as possible it is recommended that he cover Part A of Chapters 3 and 4, Sections 1-8, 12-24 of Chapter 5, and then Chapters 6 and 7.

There is an important problem concerning integration theory. Although integration theory is not needed to understand the basic material covered, there are certain examples that do make reference to the Lebesgue integral and probability spaces. This problem can be handled in at least two ways. First, it can be more or less ignored. That is, the student can be told that there is such a thing as a Lebesgue integral and what its relation to the, presumably familiar, Riemann integral is. Probability spaces can be "glossed" over in the same way. The other way to approach the problem is to use the appendices. Appendix D gives an introduction to Lebesgue integration theory, and Appendix E presents the basic facts about probability spaces.

Each chapter is denoted by a numeral; that is, Chapter 3. The tenth section of the third chapter is denoted Section 3.10. However within Chapter 3, the 3 may be dropped and Section 10 used instead of Section 3.10. Theorem 5.5.4 (or Definition 5.5.4, Lemma 5.5.4, Corollary 5.5.4) refers to the fourth theorem in Section 5 of Chapter 5.

The notation "▮" is used to denote the end of proofs and examples. This allows the proof or examples to be skimmed on first reading.

The authors would like to thank a number of people who have aided in the development of this book. First, there are the students at various universities who have taken courses from one or the other of us based upon manuscript versions. Their suggestions have been invaluable. Next, we would like to thank colleagues who have aided us in various ways: H. Antosiewicz, M. Damborg, K. Irani, G. Kallianpur, W. Kaplan, W. Littman, W. Miller, R. Perret, W. Porter, T. Pitcher, P. Rejto, Y. Sibuya, H. van Nauta Lemke, and H. Weinberger. We especially want to thank F. Beutler for the many suggestions that arose out of his classroom use of the manuscript. Finally, we would like to thank the many secretaries at various universities who have helped in the preparation of the manuscript. In particular, we would like to thank the secretarial staffs of the Department of Electrical Engineering at the University of Michigan and the School of Mathematics at the University of Minnesota.

Ann Arbor
Minneapolis
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George R. Sell

Preface to the Second Edition

We are very pleased that the new edition is being published and we are grateful to Springer-Verlag for doing this. The number of inquiries that we received each year made us believe that a new edition would be welcomed. We hope we were right, and we hope that it will be of use to our colleagues and their students.

We further hope, probably unrealistically, that we have corrected all errors of the first edition.

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George R. Sell

Contents

I

Introduction

1. BLACK BOXES

A great number of the mathematical problems of engineering and science can be fruitfully viewed as what are often referred to as "black box problems." One puts an "input" into a black box (Figure 1.1.1), the black box hums and

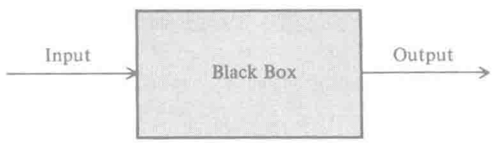

Figure 1.1.1.

whirls inside, and out comes an "output." Black box problems are questions about what black boxes do. The following are a few examples:

(1) If a black box is in fact an amplifier, questions can be asked about bandwidth, unit step response, distortion, and so on.

(2) Given an autonomous differential equation $\dot{x} = f(x)$, the initial state (that is, the initial conditions) may be viewed as the input and the resulting motion (or solution) may be viewed as the output. Many questions can be asked about the behavior of such equations; for example, questions about asymptotic growth, stability, periodicity, and so on.

(3) The input data to a digital computer is a string of symbols and its corresponding output is another string of symbols. The program determines what this black box does.

(4) Let $S = \{s_1, s_2, \ldots, s_n\}$ denote the state set for a Markov chain and $p(k)$, $k = 0, 1, 2, \ldots$, denote the probability distribution over S at time k. Further let A denote the matrix of transition probabilities; that is,

$$p(k+1) = Ap(k) \qquad k = 0, 1, 2, \ldots.$$

One can view the initial probability distribution $p(0)$ as an input and the resulting sequence $p(1), p(2), \ldots$ as the output. Or one can view $p(k)$ as an input and the resulting $p(k+1)$ as an output.

(5) In the case of a plucked string, the initial stretched position of the string before release can be viewed as an input and the resulting string vibration can be viewed as an output.

(6) In a quantum-mechanical system, the wave function $\psi(x, t)$ may be viewed as an input and the integral $\int |\psi(x, t)|^2 \, dx$ or the partial derivative $\partial\psi/\partial t$ may be viewed as the output.

Needless to say, there is no end to the problems that can be formulated as black box problems.

As far as this book is concerned, the most important aspects of black box problems are that, once surface detail is removed, seemingly different problems become similar to one another and that certain patterns repeatedly appear in solution methods. For example, one does not treat linear time-invariant network problems as separate unrelated problems, rather one approaches them as a unified class of closely related problems. Similarly, it was noticed long ago that at a certain level of abstraction the matrix equation

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & & a_{1n} \\ & \cdots & \\ & & \\ a_{n1} & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$$

and the integral equation

$$y(t) = \int_0^T k(t, \tau)x(\tau) \, d\tau \qquad t \in [0, T]$$

describe similar mathematical situations. Another way to say this is that these problems have similar mathematical structures.

The black box is thus an "operator" which transforms an input into an output. It is these operators that form the subject matter of our book. What we want to do, then, is recognize and study the essential mathematical structure of these operators. Although there are many kinds of operators, our goal here is to study those that can, once unessential details are removed, be viewed as transformations from normed linear spaces into normed linear spaces. This allows us to treat in a unified manner, matrix equations, integral equations, differential equations, difference equations, and random processes.

The real Euclidean plane is an example of a normed linear space. N-dimensional Euclidean space is another. Certain sequence spaces and function spaces are also examples. There are many other examples as we shall see later. For us the most important fact about normed linear spaces is that they all have a geometric structure that is very similar to ordinary two- or three-dimensional Euclidean geometry. This is particularly true for Hilbert spaces, a special subclass of normed linear spaces. *This geometric structure is the unifying theme of the material presented in this book.*

The first part (Chapters 2, 3, 4, and 5) of this book is devoted to a detailed study of this geometric structure. It turns out that the geometric structure of a normed linear space really involves three different kinds of structure: set-theoretic, topological, and algebraic. We illustrate this subdivision in the next section with the aid of a familiar example: the plane.

2. STRUCTURE OF THE PLANE

The real Euclidean plane is a classic example of a normed linear space. As we have noted, it has set-theoretic, topological, and algebraic structure.

Set-Theoretic Structure

Before anything else, the plane is a set. In particular, it is the set of all ordered pairs of real numbers $x = (x_1, x_2)$. Denote this set by R^2, and note that (7,1) and (1,7) are different points in this set. We refer to this set as the underlying set.

Topological Structure

The type of topological structure that we are interested in here has to do with the concept of closeness. In particular, the Euclidean distance d between any two points $x = (x_1,x_2)$ and $y = (y_1,y_2)$ is

$$d(x,y) = \{\,|x_1 - y_1|^2 + |x_2 - y_2|^2\}^{1/2}.$$

The set R^2 equipped with this distance function is an example of what is called a metric space.

Algebraic Structure

The type of algebraic structure that we are interested in here is addition and scalar multiplication of points (vectors) in the plane. Thus, if $x = (x_1,x_2)$ and $y = (y_1,y_2)$, then

$$x + y = (x_1 + y_1, x_2 + y_2).$$

And if α is any real number,

$$\alpha x = (\alpha x_1,\alpha x_2).$$

With this structure on the set R^2 we have a linear space.

Combined Topological and Algebraic Structure

It is possible to have metric spaces that are not linear spaces and vice versa. As we have just seen here, it is also possible to have both a topological and an algebraic structure on the same underlying set. It happens very often that the topological and algebraic structure are blended together. In the case of the plane, and normed linear spaces in general, this blending is accomplished by means of the norm or length of vectors in the plane. If $x = (x_1,x_2)$, then the norm of x is given by

$$\|x\| = (x_1{}^2 + x_2{}^2)^{1/2}. \tag{1.2.1}$$

It follows that

$$d(x,y) = \|x - y\|$$

and

$$\|\alpha x\| = |\alpha|\,\|x\|,$$

where α is any scalar. Neither of the above two expressions would make sense if we did not have algebraic structure. We will see later that addition and scalar multiplication have continuity properties which are also a result of the blending of topological and algebraic structure.

Geometric Structure

When we put all the pieces together we are back to the plane with its familiar geometry. Some geometric facts are the result of the presence of topological structure only, some the result of the presence of algebraic structure only, and some involve both. We shall see which are which in the following four chapters.

Before we go on, it should be noted that the norm in (1.2.1) has some additional structure, namely that it is generated by an inner product. The inner product between the two vectors x and y in R^2 is given by

$$(x,y) = x_1y_1 + x_2y_2 .$$

Thus $\|x\| = (x,x)^{1/2}$. It should be noted here that there are other norms one can prescribe on R^2 that are not generated by inner products. We shall see in Chapter 5 that the geometric structure of spaces with inner products is much richer than those without.

3. MATHEMATICAL MODELING

Since successful application of mathematics depends on successful mathematical modeling, it is worthwhile to say a few words about mathematical modeling. Roughly speaking, it is the formulation of a mathematical system whose mathematical behavior models certain aspects of a real system. For example, Ohm's law $e = Ri$ gives a mathematical model for the electrical behavior of a resistor.

The resistor can be used to illustrate the main problem of mathematical modeling. In order to formulate a mathematical model which models many aspects of a real system, one is usually led to a mathematical model of great complexity and such models are often mathematically intractable. For example, to model the high frequency as well as the high voltage behavior of our resistor could require a mathematical model involving nonlinear partial differential equations. Such equations are notoriously difficult. On the other hand, if one allows simple mathematical models only, one often ends up with a mathematical model which does not yield a sufficiently accurate or detailed description of the real system's behavior. For example, treating a long telephone or power line as a resistor without inductance and capacitance leads to a simple, yet usually inadequate mathematical model.

The formulation, then, of a mathematical model is a compromise between mathematical intractability and inadequate description of the system being modeled. There usually is a choice of mathematical models between these two extremes. For this reason, one usually talks about "a" mathematical model for a system not "the" mathematical model.

Another point to be made about mathematical modeling is that it is by no means a purely mathematical problem. It has a mathematical side, but it also has, for example, a physical or economical side. Indeed, mathematics alone would not allow us to arrive at Ohm's law. We need physics too. Mathematical modeling is the interface or bridge between pure mathematics and other disciplines.

4. THE AXIOMATIC METHOD. THE PROCESS OF ABSTRACTION

The reader probably had his first encounter with the axiomatic method in the study of Euclidean geometry. Since all of mathematics and the subject matter of this book, in particular, is based on the axiomatic method, let us recall some of the features of axiomatic reasoning.

In every branch of mathematics one starts out with a collection of "undefinables." In the Euclidean geometry (of the plane) this includes "points" and "lines." Next, certain properties are stated. These properties (axioms, postulates) play the role of mathematical legislation and form the starting point of mathematical life, or reasoning. While these axioms usually have some basis in intuition, it should be emphasized that mathematical reasoning plays no role[1] in establishing these axioms. Some of the axioms of Euclidean geometry are: (a) the parallel postulate, and (b) if L is a line, then there exists a point not on L. Once the axioms have been chosen, one then tries to prove certain properties or theorems. For example, congruence or similarity of triangles is a question studied in Euclidean geometry.

The axiomatic method is *the* method of mathematics, in fact, it is mathematics. Even though there are many controversies in the mathematical community over the contents of sets of axioms, there is no question over the role of the axioms.

While the role of the axiomatic method in mathematics has been known for centuries, the emphasis of this role that one finds today is something which developed only recently. One can see this change by comparing the research papers of the last century with those published today. In the past it required very careful reading in order to determine the hypotheses needed in order to get a particular conclusion. Today, with most papers written in the definition-theorem-proof style, it is very easy to determine this.

Of course, axiomatic systems just do not happen. They must be formulated. As mentioned in Section 1, while working on seemingly diverse problems, one often finds that similar techniques are being employed. For example, the reader may be familiar with the z-transform and Laplace-transform techniques as applied to discrete-time and continuous-time systems, respectively. Another example would be the techniques used to study the harmonics of a vibrating string and the energy levels of the hydrogen atom. It is natural, then, to inquire into the essential features (or properties) of these techniques which allow them to be applied in different ways. By listing these properties (or axioms) as hypotheses and deriving results from them one thereby goes from a concrete problem to a more abstract

[1] We should note that a set of axioms should be consistent, that is, they should not lead to contradictory statements. This question of consistency is a very important question in mathematical logic, but we shall not go into it here. Instead, we refer the reader to Wilder [1].