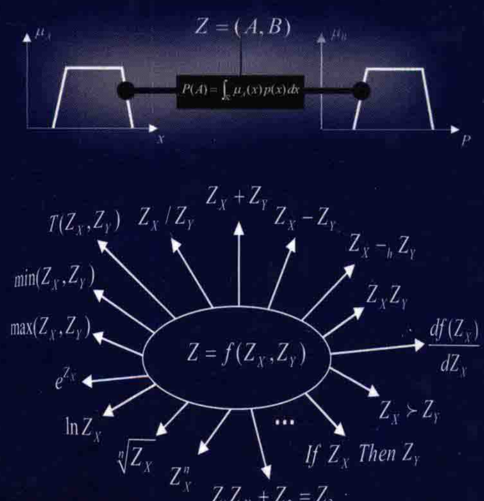
The Arithmetic of Z-Numbers

Theory and Applications



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THE ARITHMETIC OF Z-NUMBERS
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Dedication

Dedicated to the memory of my wife Aida Alieva

Rafik Aliev

To my parents and grandparents

Oleg Huseynov

To my parents and family

Rashad Aliev

To the memory of my father Vali Alizadeh

Akif Alizadeh

Preface

Real-world information is imperfect and we often use natural language (NL) in order to represent this feature of the former. On the one hand, such information is often characterized by fuzziness. This implies that we often impose soft constraints on values of variables of interest. On the other hand, what is very important is that it is not sufficient to take into account only fuzziness when dealing with real-world imperfect information. The other essential property of information is its partial reliability. Indeed, any estimation of values of interest, be it precise or soft, are subject to the confidence in sources of information we deal with – knowledge, assumptions, intuition, envision, experience – which, in general, cannot completely cover the whole complexity of real-world phenomena. Thus, fuzziness from the one side and partial reliability form the other side are strongly associated to each other. In order to take into account this fact, L.A. Zadeh suggested the concept of a Z-number as a more adequate formal construct for description of real-world information. A Z-number is an ordered pair $Z=(A,B)$ of fuzzy numbers used to describe a value of a variable X, where A is an imprecise constraint on values of X and B is an imprecise estimation of reliability of A and is considered as a value of probability measure of A.

The concept of a Z-number has a potential for many applications, especially in the realms of computation with probabilities and events described in NL. Of particular importance are applications in economics, decision analysis, risk assessment, prediction, anticipation, planning, biomedicine and rule-based manipulation of imprecise functions and relations.

Thus, real-world information is often represented in a framework of Z-number based evaluations. Such information is referred to as Z-information. The main critical problems that naturally arises in processing Z-information are computation and reasoning with Z-information. The existing literature devoted to computation with Z-numbers is quite scarce. Unfortunately, there is no general and computationally effective approach to computations with Z-numbers. There is a need in development of a universal approach to computations with Z-numbers which can be relatively easily applied for solving a wide spectrum of real-world problems in control, decision analysis, optimization and other areas. Computation and reasoning with Z-information are characterized by propagation of restrictions, that is, they are restriction-based computation and reasoning. As it is mentioned by L.A. Zadeh, the principal types of restrictions are probabilistic restrictions, possibilistic restrictions and combinations of probabilistic and possibilistic restrictions. Indeed, Z-information falls within the category of possibilistic-probabilistic restrictions. Nowadays, the existing literature devoted to computation and reasoning with restrictions includes well-developed approaches and theories to deal with pure probabilistic or pure possibilistic restrictions. For computation with probabilistic restrictions as probability distributions, the well-known probabilistic arithmetic is used. Fuzzy arithmetic deals with possibilistic constraints, which describe objects as classes with "unsharp" boundaries.

Unfortunately, up to day there is no approach to computation and reasoning with objects described by combination of probabilistic and possibilistic restrictions, such as Z-numbers. Arithmetic of Z-numbers is a basis of a future mathematical formalism to process Z-information. Arithmetic of Z-numbers is greater than just "mechanical sum" of probabilistic arithmetic and fuzzy arithmetic, it is a synergy of these two counterparts. Consequently, development of this arithmetic requires generalization of the extension principle to deal with a fusion of probabilistic and possibilistic restrictions. In turn, computation of restrictions is computation of functions and functionals that involves optimization problems, particularly, mathematical programming and variational problems.

Nowadays there is no arithmetic of Z-numbers suggested in the existing literature. The suggested book is the first to present a comprehensive and self-contained theory of Z-arithmetic and its applications. Many of the concepts and techniques described in the book are original and appear in the literature for the first time.

This book provides a detailed method in arithmetic of continuous and discrete Z-numbers. We also provide the necessary knowledge in its connections to other types of theories of uncertain computations. In addition, we discuss widely application of Z-numbers in variety of methods of operations research, economics, business and medicine.

Let us emphasize that many numbers, especially, in fields such as economics and decision analysis, are in reality Z-numbers, but they are not treated as such, because it is much simpler to compute with numbers than with Z-numbers. Basically, the concept of a Z-number is a step toward formalization of the remarkable human capability to make rational decisions in an environment of imprecision and uncertainty.

The book is organized into 7 chapters. The first chapter includes papers of L.A. Zadeh: *L.A. Zadeh. Toward a restriction-centered theory of truth and meaning (RCT). Information Sciences, 248, 2013, 1–14*; and *L.A. Zadeh, A note on Z-numbers, Information Sciences, 181, 2011, 2923-2932.* In the first section, the restriction centered theory, RCT, is considered which may be viewed as a step toward formalization of everyday reasoning and everyday discourse. Unlike traditional theories—theories which are based on bivalent logic—RCT is based on fuzzy logic. In the second section, the general concepts of a Z-number and Z^+-number are suggested which have a potential for many applications. Also, sound theoretical foundation of computation of different functions of Z-numbers are suggested.

For the present book to be self-containing, foundations of fuzzy sets theory, fuzzy logic and fuzzy mathematics which are used as the formal basis of the suggested theory of computation with Z-numbers are given in Chapter 2. We would like to mention that this chapter contains a material on a spectrum of computations with uncertain and imprecise information including the basics of interval arithmetic, probabilistic arithmetic, and fuzzy arithmetic. In this chapter we also give properties of continuous and discrete Z-numbers.

Operations on continuous Z-numbers are explained in Chapter 3. Arithmetic operations such as addition, standard subtraction, multiplication and standard division are considered. Also, square and square root of a continuous Z-number are given. Chapter 4 provides a method of computation with discrete Z-numbers. Taking into account the fact that real problems are characterized by linguistic information which is, as a rule, described by a discrete set of meaningful linguistic terms, in this book we consider also discrete Z-numbers. This chapter includes original methods for performing all arithmetic operations: addition, standard subtraction, Hukuhara difference, multiplication and standard division of discrete Z-numbers; computation methods for square of a discrete Z-number, square root of a discrete Z-number, maximum and minimum of discrete Z-numbers, and ranking of discrete Z-numbers are also suggested. Algebraic system of Z-numbers is described in Chapter 5. Here, distance between two discrete Z-numbers, functions of discrete Z-numbers, equations with discrete Z-numbers, derivative of a function of discrete Z-numbers, t-norm and t-conorm of discrete Z-numbers, aggregation of discrete Z-numbers, and functions as discrete Z-numbers-based IF-THEN rules (Z-rules) are considered. Different methods of aggregation of Z-numbers, mainly, T-norm, T-conorm, weighted average and Choquet integral-based aggregations are given. In this chapter, a special emphasis is put on Z-rules and interpolation procedures for reasoning.

All the methods in Chapters 1-5 are illustrated by a vast spectrum of examples.

Chapters 6, 7 deal with applications of Z-numbers in different areas. In Chapter 6 we suggest a new approach to solving a Z-valued linear programming problem and construction of Z-linear regression models, We also consider Z-restriction based multicriteria choice problem. A special emphasis is done on decision making under Z-information and computing with words in Z-information framework.

Chapter 7 is devoted to application of the methods suggested in Chapters 1-6 to real-world problems in economics, business and planning problems.

This book is intended to offer comprehensive coverage of the methods for computation with Z-numbers. It is written to be suitable to different groups of readers, mainly for senior college students, graduate students and for researchers and practitioners with advanced knowledge in statistics, fuzzy logic and the Z-number theory.

Our goal was set to write a complete introductory and comprehensive book on Z-number based uncertain computation. We hope that this book has led to a good foundation for learning computation with Z-numbers, as well as being a stepping stone to farther research in this new and very important theory and practice.

The book will be helpful for professionals, academicians, managers and graduate students in fuzzy logic, decision sciences, artificial intelligence, mathematical economics, and computational economics.

We would like to express our thanks to Professor Lotfi Zadeh, the founder of the fuzzy set and Soft Computing theories and a creator of the idea of Z-number, for his permanent support, invaluable ideas and advices for our research.

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Contents

Chapter 1

The General Concept of a Restriction and Z-numbers

This chapter includes the papers of L.A. Zadeh: 1. *L.A. Zadeh, Toward a restriction-centered theory of truth and meaning (RCT). Information Sciences, 248, 2013, 1–14*; 2. *L.A. Zadeh, A note on Z-numbers, Information Sciences, 181, 2011, 2923-2932.*

1.1. Z-restriction

1.1.1. *Introduction*

The concepts of truth and meaning are of fundamental importance in logic, information analysis and related fields. The restriction-centered theory outlined in this paragraph, call it RCT for short, is a departure from traditional theories of truth and meaning, principally correspondence theory, coherence theory, Tarski semantics, truth-conditional semantics and possible-world semantics [42, 43, 83, 105, 115, 119, 131, 134].

In large measure, traditional theories of truth and meaning are based on bivalent logic. RCT is based on fuzzy logic. Standing on the foundation of fuzzy logic, RCT acquires a capability to enter the realm of everyday reasoning and everyday discourse — a realm which is avoided by traditional theories of truth and meaning largely because it is a realm that does not lend itself to formalization in the classical tradition.

In RCT, truth values are allowed to be described in natural language. Examples.Quite true, very true, almost true, probably true, possibly true, usually true, etc. Such truth values are referred to as linguistic truth values. Linguistic truth values are not allowed in traditional logical systems.

The centerpiece of RCT is the deceptively simple concept—the concept of a restriction. The concept of a restriction has greater generality than the concept of interval, set, fuzzy set and probability distribution. An early discussion of the concept of a restriction appears in [153]. Informally, a restriction, *R(X)*, on a variable, *X*, is an answer to a question of the form: What is the value of *X* ? Example. Robert is staying at a hotel in Berkeley. He asks the concierge, ''How long will it take me to drive to SF Airport?'' Possible answers: 1 h, 1 h plus/minus fifteen minutes, about 1 h, usually about 1 h, etc. Each of these answers is a restriction on the variable, driving time. Another example. Consider the proposition, p: Most Swedes are tall. What is thetruth value of p? Possible answers: true, 0.8, about 0.8, high, likely high, possibly true, etc. In RCT, restrictions are preponderantly described as propositions drawn from a natural language. Typically, a proposition drawn from a natural language is a fuzzy proposition, that is, a proposition which contains fuzzy predicates, e.g., tall, fast, heavy, etc., and/or fuzzy quantifiers, e.g., most, many, many more, etc., and/or fuzzy probabilities, e.g., likely, unlikely, etc. A zero-order fuzzy proposition does not contain fuzzy quantifiers and/or fuzzy probabilities. A first-order fuzzy proposition contains fuzzy predicates and/or fuzzy quantifiers and/or fuzzy probabilities. It is important to note that in the realm of natural languages fuzzy propositions arethe norm rather than exception. Traditional theories of truth and meaning provide no means for reasoning and computation with fuzzy propositions.

Basically, *R(X)* may be viewed as a limitation on the values which *X* can take. Examples.

> $X = 5$; *X* is between 3 and 7; *X* is small; *X* is normally distributed with mean m and variance σ^2 ; It is likely that *X* is small
>
> Summers are usually cold in San Francisco
> (*X* is implicit)
>
> Robert is much taller than most of his friends
> (*X* is implicit)

As a preview of what lies ahead, it is helpful to draw attention to two key ideas which underlie RCT. The first idea, referred to as the meaning postulate, MP, is that of representing a proposition drawn from a natural language, p, as a restriction expressed as

$$p \to X \text{ isr } R,$$

where X is the restricted variable, R is the restricting relation, and r is an indexical variable which defines the way in which R restricts X. X may be an n-ary variable, and R may be an n-ary relation. Generally, X and R are implicit in p. Basically, X is the variable whose value is restricted by p. X is referred to as the focal variable. In large measure, the choice of X is subjective, reflecting one's perception of the variable or variables which are restricted by p. However, usually there is a consensus. It should be noted that a semantic network representation of p may be viewed as a graphical representation of an n-ary focal variable and an n-ary restricting relation. The expression on the right-hand side of the arrow is referred to as the canonical form of p, CF(p). CF(p) may be interpreted as a generalized assignment statement [161]. The assignment statement is generalized in the sense that what is assigned to X is not a value of X, but a restriction on the values which X can take. Representation of p as a restriction is motivated by the need to represent p in a mathematically well-defined form which lends itself to computation.

The second key idea is embodied in what is referred to as the truth postulate, TP. The truth postulate equates the truth value of p to the degree to which X satisfies R. The degree may be numerical or linguistic. As will be seen in the sequel, in RCT the truth value of p is a byproduct of precisiation of the meaning of p.

To simplify notation in what follows, in some instances no differentiation is made between the name of a variable and its instantiation. Additionally, in some instances no differentiation is made between a proposition, p, and the meaning of p.

1.1.2. *The Concept of a Restriction – A Brief Exposition*

A restriction *R(X)* on variable *X* may be viewed as information about *X*. More concretely, *R(X)* may be expressed in a canonical form, *CF(R(X)),*

$$\text{CF(R(X))} : \text{X isr R}.$$

A restriction is precisiated if *X*, *R* and r are mathematically well defined. Precisiation of restrictions plays a pivotal role in RCT. Precisiation of restrictions is a prerequisite to computation with restrictions. Here is an example of a simple problem which involves computation with restrictions.

Usually Robert leaves his office at about 5 pm.

Usually it takes Robert about an hour to get home from work.

At what time does Robert get home?

Humans have a remarkable capability to deal with problems of this kind using approximate, everyday reasoning. One of the important contributions of RCT is that RCT opens the door to construction of mathematical solutions of computational problems which are stated in a natural language.

There are many types of restrictions. A restriction is singular if *R* is a singleton. Example. $X = 5$. A restriction is nonsingular if *R* is not a singleton. Nonsingularity implies uncertainty. A restriction is direct if the restricted variable is *X*. A restriction is indirect if the restricted variable is of the form *f(X)*. Example.

$$R(p) : \int_a^b \mu(u)p(u)du \text{ is likely},$$

is an indirect restriction on p.

In the sequel, the term restriction is sometimes applied to *R*.

The principal types of restrictions are: possibilistic restrictions, probabilistic restrictions and Z-restrictions.

Possibilistic restriction (r = blank)

$$\text{R(X): X is } A,$$

where A, is a fuzzy set in a space, U, with the membership function, μ_A. A plays the role of the possibility distribution of X,

$$\text{Poss}(X = u) = \mu_A(u).$$

Example.

X is small
↑ ↑
restricted variable restricting relation (fuzzy set).

The fuzzy set small plays the role of the possibility distribution of *X* (Fig. 1.1).

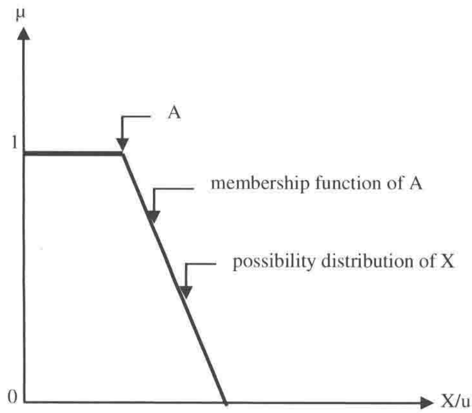

Fig. 1.1. Possibilistic restriction ox X.

Example.

Leslie is taller than Ixel →
(Height (Leslie), Height(Ixel)) is taller
↑ ↑
restricted variable restricting relation (fuzzy relation)

The fuzzy relation taller is the possibility distribution of ((Height (Leslie), Height (Ixel)).

Probabilistic restriction (r = p)

$$R(X) : X \text{ isp } p,$$

where *p* is the probability density function of *X*,

$$\text{Prob}\,(u \leq X \leq u + du) = p(u)du.$$