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Mathematical Bridges

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ISBN 978-0-8176-4394-2 ISBN 978-0-8176-4629-5 (eBook)
DOI 10.1007/978-0-8176-4629-5

Library of Congress Control Number: 2017932379

Mathematics Subject Classification (2010): 00A07, 08A50, 15-XX, 26-XX, 97H60, 97I20, 97I30

Printed on acid-free paper

This book is published under the trade name Birkhäuser, www.birkhauser-science.com
The registered company is Springer Science+Business Media LLC
The registered company address is: 233 Spring Street, New York, NY 10013, U.S.A.

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Preface

Many breakthroughs in research and, more generally, solutions to problems come as the result of someone making *connections*. These connections are sometimes quite subtle, and at first blush, they may not appear to be plausible candidates for part of the solution to a difficult problem. In this book, we think of these connections as *bridges*. A bridge enables the possibility of a solution to a problem that may have a very elementary statement but whose solution may involve more complicated realms that may not be directly indicated by the problem statement. Bridges extend and build on existing ideas and provide new knowledge and strategies for the solver. The ideal audience for this book consists of ambitious students who are seeking useful tools and strategies for solving difficult problems (many of olympiad caliber), primarily in the areas of real analysis and linear algebra.

The opening chapter (aptly called “Chapter 1”) explores the metaphor of bridges by presenting a myriad of problems that span a diverse set of mathematical fields. In subsequent chapters, it is left to the *reader* to decide what constitutes a bridge. Indeed, different people may well have different opinions of whether something is a (useful) bridge or not. Each chapter is composed of three parts: the theoretical discussion, proposed problems, and solutions to the proposed problems. In each chapter, the theoretical discussion sets the stage for at least one bridge by introducing and motivating the themes of that chapter—often with a review of some definitions and proofs of classical results. The remainder of the theoretical part of each chapter (and indeed the majority) is devoted to examining illustrative examples—that is, several problems are presented, each followed by at least one solution. It is assumed that the reader is intimately familiar with real analysis and linear algebra, including their theoretical developments. There is also a chapter that assumes a detailed knowledge of abstract algebra, specifically, group theory. However, for the not so familiar with higher mathematics reader, we recommend a few books in the bibliography that will surely help, like [5–9, 11, 12].

Bridges can be found everywhere—and not just in mathematics. One such final bridge is from us to our friends who carefully read the manuscript and made extremely valuable comments that helped us a lot throughout the making of the book. It is, of course, a bridge of acknowledgments and thanks; so, last but not least, we must say that we are deeply grateful to Gabriel Dospinescu and Chris Jewell for all their help along the way to the final form of our work.

In closing, as you read this book, we invite you to discover some of these bridges and embrace their power in solving challenging problems.

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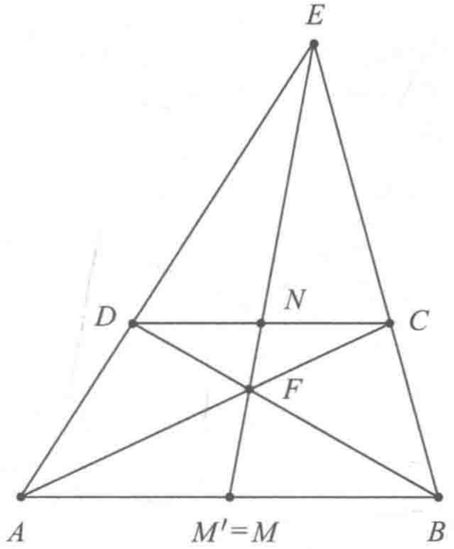
Chapter 1
Mathematical (and Other) Bridges

Many people who read this book will probably be familiar with the following result (very folkloric, if we may say so).

Problem 1. The midpoints of the bases of a trapezoid, the point at which its lateral sides meet, and the point of intersection of its diagonals are four collinear points.



Solution. Indeed, let $ABCD$ be a trapezoid with $AB \parallel CD$; let M and N be the midpoints of the line segments AB and CD, respectively; and let $\{E\} = AD \cap BC$ and $\{F\} = AC \cap BD$. We intend to prove that M, N, E, and F are four collinear points.

Denote by M' the intersection of EF with AB. By Ceva's theorem, we have

$$\frac{AM'}{M'B} \cdot \frac{BC}{CE} \cdot \frac{ED}{DA} = 1.$$

T. Andreescu et al., *Mathematical Bridges*, DOI 10.1007/978-0-8176-4629-5_1

Also, Thales' interception theorem says that

$$\frac{DE}{DA} = \frac{CE}{CB} \Leftrightarrow \frac{CB}{CE} \cdot \frac{DE}{DA} = 1,$$

and by putting together the above two equations, we get $M'A = M'B$, that is, M' is actually the midpoint of AB; therefore, $M = M'$ belongs to EF, which is (part of) what we intended to prove. □

The reader will definitely find a similar way (or will be able to use the already proved fact about the collinearity of M, E, and F) to show that N, E, and F are also collinear.

One can also prove that a converse of this theorem is valid, that is, for instance, if M, E, and F are collinear, then AB and CD are parallel (just proceed analogously, but going in the opposite direction). Or try to prove that if the midpoints of two opposite sides of a trapezoid and the intersection point of its diagonals are three collinear points, then the quadrilateral is actually a trapezoid (the sides whose midpoints we are talking about are the parallel sides); this could be more challenging to prove.

As we said, this is a well-known theorem in elementary Euclidean geometry, so why bother to mention it here? Well, this is because we find in it a very good example of a problem that needs a *(mathematical) bridge*. Namely, you noticed that the problem statement is very easy to understand even for a person who only has a very humble background in geometry—but that person wouldn't be able to *solve* the problem. You could be familiar with basic notions as collinearity and parallelism, you could know such things as properties of angles determined by two parallel lines and a transversal, but any attempt to solve the problem with such tools will fail. One needs much more in order to achieve such a goal, namely, one needs a *new theory*—we are talking about the theory of similarity. In other words, if you want to solve this problem, you have to raise your knowledge to new facts that are not mentioned in its statement. You need to throw a *bridge* from the narrow realm where you are stuck to a larger extent.

The following problem illustrates the same situation.

Problem 2. Determine all monotone functions $f : \mathbb{N}^* \to \mathbb{R}$ such that

$$f(xy) = f(x) + f(y) \text{ for all } x, y \in \mathbb{N}^*$$

($\mathbb{N}^*$ denotes the set of positive integers, while $\mathbb{R}$ denotes the reals).

Solution. All that one can get from the given relation satisfied by f is $f(1) = 0$ and the obvious generalization $f(x_1 \cdots x_k) = f(x_1) + \cdots + f(x_k)$ for all positive integers $x_1, \ldots, x_k$ (an easy and canonical induction leads to this formula) with its corollary $f(x^k) = kf(x)$ for all positive integers x and k. But nothing else can be done if you don't step into a higher domain (mathematical analysis, in this case) and if you don't come up with an idea. The idea is possible in that superior domain, being somehow natural if you want to use limits.

Let n be a positive integer (arbitrary, but fixed for the moment), and let us consider, for any positive integer k, the unique nonnegative integer n_k such that $2^{n_k} \le n^k < 2^{n_k+1}$. Rewriting these inequalities in the form

$$\frac{\ln n}{\ln 2} - \frac{1}{k} < \frac{n_k}{k} \le \frac{\ln n}{\ln 2}$$

one sees immediately that $\lim\limits_{k\to\infty} n_k/k = \ln n/\ln 2$.

Because if f is increasing, $-f$ is decreasing and satisfies the same functional equation (and conversely), we can assume, without loss of generality, that f is increasing. Then from the inequalities satisfied by the numbers n_k and by applying the noticed property of f, we obtain

$$f(2^{n_k}) \le f(n^k) \le f(2^{n_k+1}) \Leftrightarrow \frac{n_k}{k}f(2) \le f(n) \le \left(\frac{n_k}{k} + \frac{1}{k}\right)f(2).$$

Now we can let k go to infinity, yielding

$$f(n) = f(2)\frac{\ln n}{\ln 2}$$

for any positive integer n. So, all solutions are given by a formula of type $f(n) = a\ln n$, for a fixed real constant a. If f is strictly increasing (or strictly decreasing), we get $f(2) > f(1) = 0$ (respectively, $f(2) < f(1) = 0$); thus $f(2) \neq 0$, and, with $b = 2^{1/f(2)}$, the formula becomes $f(n) = \log_b n$ (with greater, respectively lesser than 1 base b of the logarithm according to whether f is strictly increasing, or strictly decreasing). The null function ($f(n) = 0$ for all n) can be considered among the solutions, if we do not ask only for strictly monotonic functions. By the way, if we drop the monotonicity condition, we can find numerous examples of functions that only satisfy the first condition. For instance, define $f(n) = a_1 + a_2 + \cdots + a_k$ for $n = p_1^{a_1} \cdots p_k^{a_k}$, with $p_1, \ldots, p_k$ distinct primes and $a_1, \ldots, a_k$ positive integers and, of course, $f(1) = 0$, and we have a function with property $f(mn) = f(m) + f(n)$ for all positive integers m and n. The interested reader can verify for himself this condition and the fact that f is not of the form $f(n) = \log_b n$, for some positive $b \neq 1$ (or, equivalently, that this function is not monotone). □

Again, one sees that in order to solve such a problem, one needs to build a bridge between the very elementary statement of the problem and the much more involved realm of mathematical analysis, where the problem can be solved.

However, there is more about this problem for the authors of this book. Namely, it also demonstrates another kind of bridge—a bridge over the troubled water of time, a bridge connecting moments of our lives. As youngsters are preoccupied by mathematics, we had (behind the Iron Curtain, during the Cold War) very few sources of information and very few periodicals to work with. There were, say, in the 80s of the former century, *Gazeta Matematică* and *Revista matematică din Timişoara*—only two mathematical magazines. The first one was a monthly magazine founded long ago, in 1895, by a few enthusiastic mathematicians and engineers among which Gheorghe Ţiţeica is most widely known. The second magazine used to appear twice a year and was much younger than its sister, but also had a national spreading due again to some enthusiastic editors. Anyway, this is all we had, and with some effort, we could also get access to Russian magazines such as *Kvant* or *Matematika v Şkole*, or the Bulgarian *Matematika*. Two of the authors of this book were at the time acquainted with problem 2 through *Revista Matematică din Timişoara*. They were high school students at that time and thoroughly followed up the problem column of this magazine, especially a "selected problems" column where they first met this problem (and couldn't solve it). The third author was the editor of that column—guess who is who! Anyway, for all three of us, a large amount of the problems in this book represent as many (nostalgic) bridges between past and present. Problem 2 is one of them, and we have many more, from which a few examples are presented below.

Problem 3. Are there continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)) + f(x) + x = 0 \text{ for every real } x?$$

Solution. No, there is no such function. The first observation is that if a function with the stated properties existed, then it would be strictly monotone. This is because such a function must be injective (the reader will immediately check that $f(x_1) = f(x_2)$ implies $f(f(x_1)) = f(f(x_2))$; therefore, by the given equation, $x_1 = x_2$). Now, injectivity and continuity together imply strict monotonicity; so if such a function existed, it would be either strictly increasing or strictly decreasing.

However, if f is strictly increasing, then $f \circ f$ and $f \circ f + f + 1_{\mathbb{R}}$ are also strictly increasing, which is impossible, because $f \circ f + f + 1_{\mathbb{R}}$ must equal the identically 0 function (by $1_{\mathbb{R}}$ we mean the identity function of the reals defined by $1_{\mathbb{R}}(x) = x$ for every real x). On the other hand, by replacing x with $f(x)$ in the given equation, we get $f(f(f(x))) + f(f(x)) + f(x) = 0$ for all x, and subtracting the original equation from this one yields $f(f(f(x))) = x$ for all x or $f \circ f \circ f = 1_{\mathbb{R}}$. This equality is a contradiction when f (and $f \circ f \circ f$ also) is strictly decreasing and the solution ends here. □

By the way, note that if $a_n, a_{n-1}, \ldots, a_0$ are real numbers such that the equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ has no real solutions, then there exists no continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $a_n f^{[n]} + a_{n-1} f^{[n-1]} + \cdots + a_0 f^{[0]} = 0$. Here $f^{[n]}$ is the nth iterate of f (with $f^{[0]} = 1_{\mathbb{R}}$), and 0 represents the identically 0 function. This was a (pretty challenging at the time) problem that we had on a test in the mentioned above eighties, on a preparation camp. A bridge, isn't it?

Problem 4. Prove that among any 79 consecutive positive integers, there exists at least one such that the sum of its digits is divisible by 13. Find the smallest 78 consecutive positive integers such that none of them has its sum of digits divisible by 13.

Solution. By $S(N)$ we will denote the sum of digits of the natural number N. We can always find among 79 consecutive natural numbers 40 of the form $100k + \overline{a0}$, $100k + \overline{a1}, \ldots, 100k + \overline{(a+3)9}$, with k a natural number and $a \leq 6$ a digit. Among the sums of digits of these numbers, there are $S(k)+a, S(k)+a+1, \ldots, S(k)+a+12$, that is, there are 13 consecutive natural numbers, one of which has to be divisible by 13.

Now, for the second part, we have to choose the desired numbers in such a way that no forty of them starting with a multiple of 10 are in a segment of natural numbers of the form $\{100k, 100k + 1, \ldots, 100k + 99\}$. This can only happen if the numbers are of the form $100a - 39, 100a - 38, \ldots, 100a + 38$, for some natural number a. Actually we will consider numbers of the form $10^b - 39, 10^b - 38, \ldots, 10^b + 38$, with $b \geq 2$, because it will be important how many nines there are before the last two digits. The sums of digits of the numbers $10^b, 10^b + 1, \ldots, 10^b + 38$ will cover all possibilities from 1 to 12. The sums of digits of the numbers $10^b - 39, 10^b - 38, \ldots, 10^b - 1$ will range from $9(b-2)+7$ to $9(b-2)+18$, and it is necessary that they cover exactly the same remainders modulo 13 (from 1 to 12). For this to happen, we need to have $9(b-2) + 7 \equiv 1 \pmod{13}$, which gives $b \equiv 10 \pmod{13}$. So, the smallest possible 78 such numbers are those obtained for $b = 10$, thus the (78 consecutive) numbers from 9999999961 to 10000000038. □

This is a problem that we know from the good old RMT.

Problem 5 (Erdős-Ginzburg-Ziv theorem). Prove that among any $2n - 1$ integers, one can find n with their sum divisible by n.

Solution. This is an important theorem, and it opened many new approaches in combinatorics, number theory, and group theory (and other branches of mathematics) in the middle of the twentieth century (it has been proven in 1961). However, we first met it in *Kvant*, with no name attached, and it was also *Kvant* that informed us about the original proof. Seemingly the problem looks like that (very known) one which states that from any n integers, one can choose a few with their sum divisible by n. The solution goes like this. If the numbers are $a_1, \ldots, a_n$, consider the n numbers $a_1, a_1+a_2, \ldots, a_1+a_2+\cdots+a_n$. If there is any of them divisible with n, the solution ends; otherwise, they are n numbers leaving, when divided by n, only $n-1$ possible remainders (the nonzero ones); therefore, by the pigeonhole principle, there are two of them, say $a_1 + \cdots + a_i$ and $a_1 + \cdots + a_j$, with, say, $i < j$ that are congruent modulo n. Then their difference $a_{i+1} + \cdots + a_j$ is, of course, divisible by n (and is a sum of a few of the initial numbers). We put here this solution (otherwise, we are sure that it is well-known by our readers) only to see that there is no way to use its idea for solving problem 5 (which is a much deeper theorem). Indeed, the above solution allows no control on the number of elements in the sum that results to be divisible by n; hence, it is of no use for problem 5. The proof that we present now (actually the original proof of the three mathematicians) is very ingenious and, of course, builds a bridge.

The first useful observation is that the property from the theorem is multiplicative, that is, if we name it $P(n)$, we can prove that $P(a)$ and $P(b)$ together imply $P(ab)$. This permits an important reduction of the problem to the case of prime n (and it is used in all the proofs that we know). We leave this as an (easy and nice) exercise for the reader. So, further, we only want to prove (and it suffices, too) that from any $2p - 1$ integers one can always choose p with their sum divisible by p, where p is a positive prime.

The bridge we throw is towards the following:

Theorem. *Let A and B be subsets of $\mathbb{Z}_p$, with p prime, and let*

$$A + B = \{a + b \mid a \in A,\ b \in B\}.$$

Then we have $|A + B| \geq \min\{p, |A| + |B| - 1\}$. (By $|X|$, we mean the number of elements of the set X.)

We skip the proof of this (important) theorem named after Cauchy and Davenport (the second rediscovered it a century after the first one; each of them needed it in his research on other great mathematical results), but we insist on the following:

Corollary. *Let $A_1, \ldots, A_s$ be 2-element subsets of $\mathbb{Z}_p$. Then*

$$|A_1 + \cdots + A_s| \geq \min\{p, s + 1\}.$$

In particular, if $A_1, \ldots, A_{p-1}$ are subsets with two elements of $\mathbb{Z}_p$, then

$$A_1 + \cdots + A_{p-1} = \mathbb{Z}_p;$$

that is, every element from $\mathbb{Z}_p$ can be realized as a sum of elements from $A_1, \ldots, A_{p-1}$ (one in each set).

This corollary is all one needs to prove Erdős-Ginzburg-Ziv's theorem, and it can be demonstrated by a simple induction over s. The base case $s = 1$ being evident, let's assume that the result holds for s two-element subsets of $\mathbb{Z}_p$ and prove it for $s + 1$ such subsets $A_1, \ldots, A_{s+1}$. If $s + 1 \geq p$, we have nothing to prove; hence, we may assume that the opposite inequality holds. In this case, by the induction hypothesis, there are at least $s + 1$ distinct elements $x_1, \ldots, x_{s+1}$ in $A_1 + \cdots + A_s$. Let $A_{s+1} = \{y, z\}$; then the set $A_1 + \cdots + A_s + A_{s+1}$ surely contains $x_1 + y, \ldots, x_{s+1} + y$ and $x_1 + z, \ldots, x_{s+1} + z$. But the sets $\{x_1 + y, \ldots, x_{s+1} + y\}$ and $\{x_1 + z, \ldots, x_{s+1} + z\}$ cannot be equal, because in that case, we would have

$$(x_1 + y) + \cdots + (x_{s+1} + y) = (x_1 + z) + \cdots + (x_{s+1} + z),$$

which means $(s + 1)y = (s + 1)z$. As $1 \leq s + 1 \leq p - 1$, this implies $y = z$ in $\mathbb{Z}_p$, which is impossible (because y and z are the two distinct elements of A_{s+1}). Consequently, among the elements $x_1 + y, \ldots, x_{s+1} + y$ and $x_1 + z, \ldots, x_{s+1} + z$ of $A_1 + \cdots + A_{s+1}$, there are at least $s + 2$ mutually distinct elements, finishing the proof.

Now for the proof of Erdős-Ginzburg-Ziv theorem, consider $a_1 \le a_2 \le \dots \le a_{2p-1}$ to be the remainders of the given $2p-1$ integers when divided by p, in increasing order. If, for example, $a_1 = a_p$, then $a_1 = a_2 = \dots = a_p$, and the sum of the p numbers that leave the remainders $a_1, \dots, a_p$ is certainly divisible by p; similarly, the problem is solved when any equality $a_j = a_{j+p-1}$ holds (for any $1 \le j \le p$). Thus we can assume further that (for every $1 \le j \le p$) a_j and a_{j+p-1} are distinct. Now we can consider the two-element subsets of $\mathbb{Z}_p$ defined by $A_j = \{a_j, a_{j+p-1}\}$, $1 \le j \le p-1$. (We do not use a special notation for the residue class modulo p of the number x, which is also denoted by x.)

According to the above corollary of the Cauchy-Davenport theorem, $A_1 + \cdots + A_{p-1}$ has at least p elements; therefore, it covers all $\mathbb{Z}_p$. Consequently, there exist $i_1, \dots, i_{p-1}$ such that i_j is either j or $j+p-1$ for any $j \in \{1, \dots, p-1\}$ and $a_{i_1} + \cdots + a_{i_{p-1}} = -a_{2p-1}$ in $\mathbb{Z}_p$. This means that the sum $a_{i_1} + \cdots + a_{i_{p-1}} + a_{2p-1}$ (where, clearly, all indices are different) is divisible by p, that is, the sum of the corresponding initial numbers is divisible by p, finishing the proof. □

One can observe that the same argument applies to prove the stronger assertion that among any $2p-1$ given integers, there exist p with their sum giving any remainder we want when divided by p. Also, note that the numbers $0, \dots, 0, 1, \dots, 1$ ($n-1$ zeros and $n-1$ ones) are $2n-2$ integers among which one cannot find any n with their sum divisible by n (this time n needs not be a prime). Thus, the number $2n-1$ from the statement of the theorem is minimal with respect to n and the stated property.

There are now many proofs of this celebrated theorem, each and every one bringing its amount of beauty and cleverness. For instance, one of them uses the congruence

$$\sum_{1 \le i_1 < \cdots < i_p \le 2p-1} (x_{i_1} + \cdots + x_{i_p})^{p-1} \equiv 0 \pmod{p}$$

(the sum is over all possible choices of a subset of p elements of the set $\{1, \dots, 2p-1\}$; in other words, it contains all sums of p numbers among the $2p-1$ given integers, which we denoted by $x_1, \dots, x_{2p-1}$). Knowing this congruence and Fermat's Theorem, one gets $N \equiv 0 \pmod{p}$, where N means the number of those sums of p of the given $2p-1$ integers that are not divisible by p. However, if all the possible sums weren't divisible by p, we would have $N = \binom{2p-1}{p-1} \equiv 1 \pmod{p}$, a contradiction—hence there must exist at least one sum of p numbers that is divisible by p.

This proof is somehow simpler than the previous one, but it relies on the above congruence, which, in turn, can be obtained from the general identity

$$\sum_{S \subseteq \{1,\dots,m\}} (-1)^{m-|S|} \left(\sum_{i \in S} x_i \right)^k = 0,$$

valid for all elements $x_1, \ldots, x_m$ of a commutative ring and for any $1 \le k \le m-1$. For $k = m$, we need to replace the 0 from the right hand side with $m!x_1 \ldots x_m$, and one can find results for the corresponding sum obtained by letting $k = m+1$, $k = m+2$, and so on, but this is not interesting for us here. Let us only remark how another bridge (a connection between this identity and the Erdős-Ginzburg-Ziv theorem) appeared, seemingly out of the blue. The reader can prove the identity for himself (or herself) and use it then for every group of p of the given $2p-1$ integers, with exponent $p-1$, and then add all the yielded equalities; then try to figure out (it is not hard at all) how these manipulations lead to the desired congruence and, finally, to the second (very compact) proof of the Erdős-Ginzburg-Ziv's theorem. However, we needed a bridge. What this book tries to say is that there are bridges everywhere (in mathematics and in the real life). At least nostalgic bridges, if none other are evident.

Let us see now a few more problems whose solutions we'll provide after the reader has already tried (a bit or more) to solve independently. As the whole book, the collection is eclectic and very subjective—and it is based on the good old sources from our youth, such as *Gazeta Matematică* (GM), *Revista matematică din Timişoara* (RMT), *Kvant*, the Romanian olympiad or TSTs, and so on. Most of the problems are folklore (and their solutions, too), but they first came to us from these sources. When the problems have proposers we mention them; otherwise, as they can be found in many books and magazines, we avoid any references—every reader, we are sure, knows where to find them.

Proposed Problems

1. (Mihai Bălună, RMT) Find all positive integers n such that any permutation of the digits of n (in base ten) produces a perfect square.
2. Let $a_1, \ldots, a_n$ be real numbers situated on a circumference and having zero sum. Prove that there exists an index i such that the n sums a_i, $a_i + a_{i+1}$, …, $a_i + a_{i+1} + \cdots + a_{i+n-1}$ are all nonnegative. Here, all indices are considered modulo n.
3. Prove that there exist integers a, b, and c, not all zero and with absolute values less than one million, such that $|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}$.
4. Prove that, for any positive integer k, there exist k consecutive natural numbers such that each of them is not square-free.
5. Find the largest possible side of an equilateral triangle with vertices within a unit square. (The vertices can be inside the square or on its boundary.)
6. Let A and B be square matrices of the same order such that $AB - BA = A$. Prove that $A^mB - BA^m = mA^m$ for all $m \in \mathbb{N}^*$ and that A is nilpotent.
7. (Dorel Miheţ, RMT) Prove that from the set $\{1^k, 2^k, 3^k, \ldots\}$ of the powers with exponent $k \in \mathbb{N}^*$ of the positive integers, one cannot extract an infinite arithmetic progression.

8. Let $1, 4, 8, 9, 16, 27, 32, \ldots$ be the sequence of the powers of natural numbers with exponent at least 2. Prove that there are arbitrarily long (nonconstant) arithmetic progressions with terms from this sequence, but one cannot find such a progression that is infinite.
9. (Vasile Postolică, RMT) Let $(a_n)_{n\geq 1}$ be a convergent increasing sequence. Prove that the sequence with general term

$$(a_{n+1} - a_n)(a_{n+1} - a_{n-1}) \ldots (a_{n+1} - a_1)$$

is convergent, and find its limit.

What can we say if we only know that $(a_n)_{n\geq 1}$ is increasing?
10. Let f be a continuous real function defined on $[0, \infty)$ such that $\lim_{n\to\infty} f(nt) = 0$ for every t in a given open interval (p, q) $(0 < p < q)$. Prove that $\lim_{x\to\infty} f(x) = 0$.
11. (Mihai Onucu Drimbe, GM) Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y + z) + f(x) + f(y) + f(z) = f(x + y) + f(x + z) + f(y + z)$$

for all $x, y, z \in \mathbb{R}$.
12. (Dorel Miheţ, RMT) Let $f : [a, b] \to [a, b]$ (where $a < b$ are real numbers) be a differentiable function for which $f(a) = b$ and $f(b) = a$. Prove that there exist $c_1, c_2 \in (a, b)$ such that $f'(c_1)f'(c_2) = 1$.
13. Evaluate

$$\int_0^{\pi/2} \frac{1}{1 + (\tan x)^{\sqrt{2}}} dx.$$

14. Show that

$$\lim_{n\to\infty} \int_a^b \left(1 + \frac{x}{n}\right)^n e^{-x} dx = b - a$$

for all real numbers a and b.

Solutions

1. Only the one-digit squares (that is, 1, 4, and 9) have (evidently) this property. Suppose a number with at least two digits has the property. It is well known that a number with at least two digits and for which all digits are equal cannot be a square; therefore, there must be at least two distinct digits, say a and b, with $a < b$. Then if $\overline{\ldots ab} = k^2$ and $\overline{\ldots ba} = l^2$, we clearly have $k < l$, hence $l \geq k + 1$, and

$$2k + 1 = (k + 1)^2 - k^2 \leq l^2 - k^2 = 9(b - a) \leq 81.$$