THE CLASSICAL THEORY OF ELECTRICITY AND MAGNETISM

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PREFACE

The present work is a translation of the eighth German edition. The notation for vectors has been modified, so as to bring it into line with that customary in English books. A new feature is the collection of problems and solutions. Special thanks are due to the translator for the capable and careful way in which he has carried out his task.

R. BECKER.

BERLIN, *May*, 1932.

The English publishers wish to express their thanks to the authorities of the University of London for permission to use their examination papers in the collection of examples.

PREFACE

TO THE EIGHTH GERMAN EDITION

The work entitled *An Introduction to Maxwell's Theory*, by A. Föppl, appeared in 1894. Ten years later, the second edition, recast and thoroughly revised, was published as the first volume of Max Abraham's *Theory of Electricity*. For a whole generation of physicists after that date, "Abraham-Föppl" was more widely used than any other textbook introductory to electrical theory. The fact that as many as seven editions appeared in Abraham's lifetime is convincing evidence of the estimation in which the work was held by teachers and students.

In the new edition, I have felt bound to preserve the essential features of a book so obviously suited to its purpose, and many passages have been taken over unchanged. At the same time, some fairly extensive alterations have been made in particular sections, always in the direction of laying greater emphasis on the concrete physical content of the theory, and less on its purely formal aspects. To assist the student towards a vivid comprehension of the text, the number of diagrams has been increased more than fivefold.

New sections have been added dealing with electrostriction, and with the thermodynamics of the field. The theory of the skin effect has been amplified, and the theory of waves in wires has been extended to the case when resistance is taken into account. In the exposition of the theory of alternating currents, advantage has been taken of the vector diagram used by electrical engineers. The treatment of electric currents as a cyclic system has been omitted altogether. The substance of the last two sections of the previous edition—on ferromagnetism and induction phenomena in moving bodies—has been incorporated in other sections.

In the choice of units I have followed Abraham's last edition in every detail. The system used throughout is the Gaussian system, in which the energy density in a vacuum is equal to

$$\frac{1}{8\pi}\left(\mathbf{E}^2 + \mathbf{H}^2\right) \text{ ergs/cm.}^3,$$

and the dielectric constant and permeability of a vacuum are each taken as unity. It does not seem possible at present to set up a system of units which will satisfy the electrical engineer and the physicist alike. With regard to Maxwell's theory, the difference between the physicist and the electrician is not a matter of notation merely, but of principle. The technical view adheres much more strictly than current physics does to the original form of the Faraday-Maxwell theory. The engineer looks upon the vectors **E** and **D**—even in a vacuum—as magnitudes of quite different kinds, related to one another more or less like tension and extension in the theory of elasticity. From this point of view it must of course seem a very questionable procedure, in an exposition of fundamental principles, to put the factor of proportionality K, in the equation $\mathbf{D} = K\mathbf{E}$, equal to 1 for empty space, thus artificially attributing to **D** and **E** the same dimensions. On the other hand, the distinction in principle between **D** and **E**, which is closely connected with the mechanical theory of the æther, has been absolutely abandoned in modern physics, the electromagnetic conditions at any point in empty space being now regarded as completely defined when we are given *one* electric vector **E** and *one* magnetic vector **B** (or **H**). The numerical identity of **E** and **D** (for empty space) in the Gaussian system of units is not, for the physicist, the result of an arbitrary definition, but the expression of the fact that **E** and **D** are actually the same thing. The introduction by the engineer of a dielectric constant and permeability not equal to 1 in a vacuum seems to the physicist to be merely an artifice, by means of which formulæ are reduced to a shape which is convenient for practical calculations.

For purposes of reference, a list of important formulæ is given in an appendix.

R. BECKER.

BERLIN, *February*, 1930.

INTRODUCTION

The theory of electric and magnetic phenomena, as it existed before Maxwell, was based on the conception of action at a distance between bodies which are electrified, magnetized, or traversed by electric currents. The only physicist who took a different view was Faraday. But he was not enough of a mathematician to express his ideas in the complete and self-consistent form which would have raised them to the rank of a theory. His method of regarding and describing electrical phenomena was, it is true, a mathematical one, but he did not express himself in terms of the ordinary symbolism of mathematics. This was first done by Maxwell, who translated Faraday's ideas into strict mathematical form, and thus built up a theory which differed essentially from the theory of action at a distance even in its foundations, and still more in its higher developments.

The discoveries of Heinrich Hertz supplied the proof that electromagnetic processes do actually take place in dielectrics, and in particular in free space, and the fundamental ideas of Maxwell's theory have now been accepted by all physicists.

What are the essential characteristics which distinguish Maxwell's theory of field action from the theories of action at a distance?

The essential ideas underlying Maxwell's theory which we shall have to consider are these:

1. The idea that all electric and magnetic action of one body on another separated from it is transmitted through the intervening space, whether that be empty or occupied by matter.

2. That the seat of electric or magnetic energy is to be found not only in the body which is electrified or magnetized, or which is traversed by a current, but also, and to a far greater extent, in the surrounding field.

3. That the electric current in an unclosed conducting circuit is closed, or made complete, by a supplementary "displacement current" in the dielectric, and that this displacement current is connected with the magnetic field strength in the same way as the conduction current.

4. That the flux of magnetic induction has no sources, or, in other words, that "true" magnetism is never found.

5. That light waves are electromagnetic waves.

Maxwell himself stated his equations in terms of quaternions, but only rather incidentally; in essence, his exposition is based on Cartesian methods. With the latter, however, it is difficult to grasp the connexion of the formulæ as a whole. It is much easier to do so when the vector calculus is employed. The trouble it costs to make oneself familiar with vector methods is amply repaid by the advantages gained. The use of vectors is in fact indispensable, if what we aim at is to secure as faithful a reproduction as possible of Faraday's idea of the flux of force. The theory of vectors and vector fields is therefore placed at the head of the present work. The notation is that now used by nearly all writers who are doing original work in electrodynamics. In the following chapters, the method of vectors, which is useful in rigid dynamics and in hydrodynamics as well as in electricity, will be employed throughout.

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PART I
VECTORS AND VECTOR FIELDS

CHAPTER I
Vectors

1. Definition of a Vector.

The equations of physics are ultimately relations between quantities which are immediately measurable. What a measurement tells is the number of times a given unit is contained in the quantity measured. The unit may be chosen arbitrarily (e.g. a metre, a second, a degree Centigrade), or it may be reduced to other units, previously defined, with which it is connected by an equation. The formula for the unit, obtained by solving this equation, represents the "dimensions" of the unit with respect to the other units. The so-called "absolute" system of measurement employs the three fundamental units of length, mass, and time; but no matter what units are chosen to serve as the foundation of the absolute system, the two sides of any equation in physics must "balance", i.e. they must agree with each other not only numerically but also in dimensions. In fact, if there were any disparity in the dimensions, a change in the fundamental units would destroy the numerical equality of the two sides of the equation. The fact that the dimensions must balance is taken advantage of in physical calculations as a first check upon the accuracy of an equation.

Physical quantities of the simplest type are completely defined by the assignment of a single number, along with a known unit. Such quantities are called *scalars*; mass and temperature are examples.

But there are other physical quantities, which do not belong to the class of scalars. Thus, in order to specify the final position of a point which is displaced from a given initial position, three numbers are required, say, for example, the Cartesian co-ordinates of the final point with respect to axes through the initial point. In this case we might, without introducing any new kind of quantity, work throughout with scalars, viz. the component displacements. But if we did so we should in the first place be neglecting the fact that, physically speaking, a displacement is a single idea; and, secondly, we should be importing a foreign element into the question, viz. the co-ordinate system, which has nothing to do with the displacement itself. We shall therefore introduce displacements as quantities of a new type, and establish a system of rules for their use. Only when we come to evaluate formulæ numerically will it be necessary to bring in a definite co-ordinate system.

Rectilinear displacements of a point, and all physical quantities which can be represented by such displacements (in the same way as the values of a scalar can be represented by the points of a straight line), and which also obey the same law of addition as the corresponding displacements, are called *vectors*.

A rigorous test for determining whether a quantity is a vector or not will be given in § 3 (p. 6).

2. Addition and Subtraction of Vectors.

In the definition of a vector just given, vector addition has been reduced to composition of rectilinear displacements. Take now two vectors **A*** and **B** of the same dimensions and type; in order to add them, consider a movable point situated to begin with at 1 (fig. 1). Let this point be given, first, the displacement (1, 2), representing the vector **A** in magnitude, direction, and sense; then the displacement (2, 3), agreeing in length, direction, and sense† with the vector **B**; the result is equivalent to a displacement of the movable point from 1 to 3. This rectilinear displacement which takes the point directly from 1 to 3 is called the *resultant* or *geometric sum* of the two displacements (1, 2) and (2, 3). It represents a vector **C** which, in accordance with the definition of § 1, we call the resultant or sum of the vectors **A** and **B**:

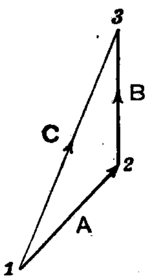

Fig. 1

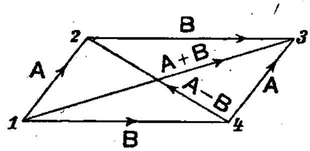

Fig. 2

$$\mathbf{C} = \mathbf{A} + \mathbf{B}. \quad \dots\dots\dots \quad (1)$$

If the displacement **B** is made first, and then the displacement **A** (fig. 2), the movable point describes the path (143), which with (123) makes up a parallelogram; accordingly the resultant of the displacements **B** and **A**, like that of **A** and **B**, is represented by the diagonal (1, 3) of that parallelogram. Hence vector addition obeys the *commutative law*: *the geometric sum of two vectors is independent of the order of addition*:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}. \quad \dots\dots\dots \quad (2)$$

* Heavy type will be used throughout to indicate vectors.

† In future, the word *direction* will be used so as to include *sense*. [Tr.]

This parallelogram law of addition (fig. 2) is characteristic of the quantities called vectors. Quantities exist with which we can associate the properties of magnitude and direction, but which follow another law of composition. For example, we know from kinematics that infinitely small rotations of a rigid system about a fixed point can be represented by vectors, since their composition obeys the parallelogram law. On the other hand finite rotations are compounded in a more complicated way, and therefore are not vectors. It is proved in statics that forces acting on a particle follow the parallelogram law of addition; such forces are therefore vectors.

If we consider the displacements derived by addition from three vectors **A**, **B**, and **C**, we see that the following law holds, called the *associative law of vector addition*:

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}). \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In fig. 3 the sum of three vectors is found by completing the quadrilateral, which has the individual vectors and their sum for its sides; and similarly the sum of n vectors is formed by means of the so-called vector polygon; this has $n + 1$ sides, namely the n vectors which are to be added, and their resultant.

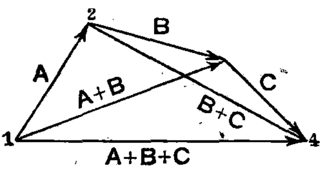

Fig. 3

We may now ask the question: What meaning is to be attached to the *geometric difference* of two vectors **A** and **B**? The answer is that we define the difference in such a way that the relation

$$\mathbf{B} - \mathbf{B} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

holds for vectors, just as the similar relation holds for scalars. The vector $-\mathbf{B}$ therefore corresponds to a displacement which annuls the displacement **B**, i.e. brings the movable point back to its original position. Thus $-\mathbf{B}$ is a vector of the same magnitude as **B**, but in the opposite direction. By the geometric difference of the vectors **A** and **B** we mean the geometric sum of **A** and $-\mathbf{B}$, so that we define *vector subtraction* as follows:

A vector **B** *is subtracted from a vector* **A** *by adding to* **A** *a vector of the same magnitude as* **B**, *but in the opposite direction.*

In the parallelogram of fig. 2, the diagonal (13) represents the geometric sum $\mathbf{A} + \mathbf{B}$, the diagonal (42) the geometric difference $\mathbf{A} - \mathbf{B}$.

The rules for the addition and subtraction of vectors which have now been laid down agree formally with the laws of ordinary algebra.

3. Unit Vectors and Fundamental Vectors. Components.

By the *product* **A** *of a scalar a and a vector* **a**,

$$\mathbf{A} = a\mathbf{a} = \mathbf{a}a, \quad \ldots \quad (5)$$

we understand a vector whose magnitude is equal to the product of the magnitudes of the scalar a and the vector **a**,

$$|\mathbf{A}| = |a| \cdot |\mathbf{a}|, \quad \ldots \quad (5a)$$

and which has the same direction as **a**, or the opposite direction, according as the scalar a is positive or negative.

The multiplication of vectors by scalars obeys the rules of the algebra of scalar quantities. The commutative law has already been explicitly stated in (5); and the distributive law also holds, i.e.

$$(\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}, \quad \alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}. \quad \ldots \quad (5b)$$

All vectors **A** which have the same direction can be connected with a vector **s**, also in that direction, and of magnitude 1:

$$\mathbf{A} = |\mathbf{A}|\mathbf{s}. \quad \ldots \quad (6)$$

A vector **s**, of magnitude 1, is called a *unit vector*. We shall adopt the convention of associating the dimensions (§ 1) of a vector with its magnitude; the unit vector **s** in (6) must therefore be given the dimensions of a pure number. Unit vectors afford a convenient means of specifying the direction of a vector, or of a number of parallel vectors.* Let there now be given a fixed unit vector **s**, and an arbitrary vector **a**, which makes with **s** the angle ϕ. The quantity

$$a_s = |\mathbf{a}| \cos\phi \quad \ldots \quad (7)$$

is called the *component of* **a** *relative to the unit vector* **s**, or the *component of* **a** *in the direction* **s**; it is equal to the length of the projection of **a** on the line of the unit vector **s**, taken with the positive or negative sign according as the projection agrees in sense with **s** or not.

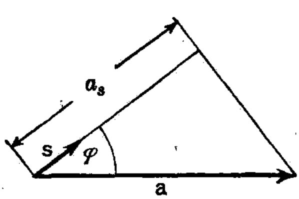

Fig. 4

The component of a vector is a scalar quantity; if we wish to express the projection of **a** on the line of the unit vector **s** in a form which indicates its direction also, we have to multiply the component of **a** in the direction **s** by the unit vector **s** itself: hence the projection as a vector is represented (fig. 4) by

$$|\mathbf{a}| \cos\phi \,.\, \mathbf{s}.$$

* The word *direction* is sometimes used as equivalent to *unit vector*. [Tr.]