Finite Element Method and its Applications

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Preface

During the past decades, giant needs for ever more sophisticated mathematical models and increasingly complex and extensive computer simulations have arisen. In this fashion, two indissoluble activities, mathematical modeling and computer simulation, have gained a major status in all aspects of science, technology, and industry. Since the partial differential equations are very important in the theory and practice, mathematicians, physicians and scientists and engineer pay greatly attention to its numerical computations. In this background, Finite element method, as numerical computational methods for solving partial differential equations is established to occur in last century. Within only a few decades, this technique has evolved from one with initial applications in structural engineering to a widely utilized and richly varied computational approach for many scientific and technological areas.

Finite Element; perhaps no other family of approximation methods has had a greater impact on the theory and practice of numerical methods during the twentieth century. Finite element methods have now been used in virtually every conceivable area of engineering that can make use of models of nature characterized by partial differential equations. There are dozens of textbooks, monographs, handbooks, memoirs, and journals devoted to its further study; numerous conferences, symposia, and workshops on various aspects of finite element methodology are held regularly throughout the world. There exist easily over one hundred reference on finite elements today, and this number is growing exponentially with further revelations of the power and versatility of the method. Today, finite element methodology is making significant inroads into fields in which many thought were outside its realm, for example computational fluid dynamics. In time, finite element methods may assume a position in this area of comparable or greater importance than classical finite difference schemes which have long dominated the subject.

As well known, the nature phenomenon can be described by instantaneous and local or global and process methodology. Two difference methodologies describe the same nature phenomenon. A main task of computational mathematics is making transformation from infinite dimensional space into an finite dimensional space, transform a continuum problem into a finite dimensional system of a discrete structure. Finite element is that a field function of infinite dimensional continuum is instead by a finite dimensional system consists of a piecewise polynomials. This means that the base functions of finite element subspace are a system of functions with finite support. Variational formulation shows the relationship any different two local points, an energy of each other acting from two different field functions which can be described by bilinear form $B(\varphi_i, \varphi_j)$ where φ_i, φ_j are based functions associated with different point.

The first part(chapter 1-3) of this book provides computational aspect of the method, second part(chapter 4,5), the mathematical background to the finite element methods and the mathematical fundamental to the method are explored, third part (chapter 6) provides nonstandard finite element method, fourth part (chapter 7-9) is concerning applications to elastic mechanics, fluid mechanics, electro-magnetic field and some engineering problems. The book should not only provide mathematical aspect, computational construct of the method and engineering application, but also it should provide a useful starting point for further research.

Contents

Chapter 1

The Structure of Finite Element Method

The finite element method is a numerical computational method for differential equations and partial differential equations. In order to solve the general field problem by using finite element method, it must pass through the following processes:

1) Find the variational formulation associated with original field problem.

2) Establish finite element subspace. For example, select the element type and associated phase functions.

3) Establish element stiffness matrix, element column and assemble global stiffness matrix-full column.

4) Treatment of the boundary conditions and solving of the system of finite element equations.

5) Come back to the real world.

In this book, the first four processes will be systematic formulations in the first chapter till third chapter.

1.1 Galerkin Variational Principle and Ritz Variational Principle

As an example, we consider the linear elliptic boundary value problem of two dimension,

$$\begin{cases} -\left[\dfrac{\partial}{\partial x}\left(p(x,y)\dfrac{\partial u}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(p(x,y)\dfrac{\partial u}{\partial y}\right)\right]=f(x,y), \\ u|_{\Gamma_1}=0, \\ \left[p(x,y)\dfrac{\partial u}{\partial n}+\sigma(x,y)u\right]_{\Gamma_2}=g(x,y), \end{cases} \tag{1.1.1}$$

where, Ω is a connected domain in R^2, $\partial\Omega=\Gamma_1\cup\Gamma_2$ is a piecewise smooth boundary. Let $\boldsymbol{n}$ denote the unit outward normal vector to $\partial\Omega$ defined almost everywhere on $\partial\Omega$. $p(x,y)\in C^1(\Omega)$, $p(x,y)\geqslant p_0>0$, $\sigma(x,y)\in C^0(\Omega)$ and $\sigma(x,y)\geqslant 0$.

Throughout this chapter we make notation: $C^0(\Omega)$ = the set of all continuous function in an open subset in R^n. $C^k(\Omega)$ = the set of functions $v\in C^0(\Omega)$, whose derivatives of order $\leqslant k$, exist and are continuous;

$$D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n},\quad D_j=\frac{\partial}{\partial x^j},\quad D_j^0=\text{identity},$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Assume that $u(x, y) \in C^2(\Omega)$ satisfies (1.1.1) in Ω and on $\partial\Omega$, the function $u(x, y)$ is called *classical solution* of problem (1.1.1).

Next, we consider weak solution of (1.1.1). Define the norm

$$\|u\|_{1,\Omega}^2 = \iint_\Omega (u_x^2 + u_y^2 + u^2)dxdy. \tag{1.1.2}$$

Sobolev space $H^1(\Omega)$ is a closure of $C^\infty(\Omega)$, under the norm (1.1.2) with the inner product

$$(u, v)_1 = \iint_\Omega (u_x v_x + u_y v_y + uv)dxdy, \tag{1.1.3}$$

$H^1(\Omega)$ is a Hilbert space which is called *one order Sobolev space.* Let

$$C_0^\infty(\Omega) = \{v : v \text{ is an infinite differentiable function and support of } v \subset \Omega\},$$

$$H_0^1(\Omega) = \text{the closure of } C_0^\infty(\Omega) \text{ under the norm (1.1.2)},$$

it is equivalent to

$$H_0^1(\Omega) = \{v : v \in H^1(\Omega),\ v|_{\partial\Omega} = 0\}.$$

In addition, let

$$C_\#^\infty(\Omega) = \{v : v \in C^\infty(\Omega),\ v|_{\Gamma_1} = 0\},$$

$$V(\Omega) = \text{closure of } C_\#^\infty(\Omega) \text{ under the norm (1.1.2)},$$

which is equivalent to

$$V = \{v : v \in H^1(\Omega),\ v|_{\Gamma_1} = 0\}.$$

It is clear that V is a Hilbert space with inner product (1.1.3). Furthermore,

$$H_0^1(\Omega) \subset V \subset H^1(\Omega).$$

Let us introduce bilinear functional

$$B(u, v) = \iint_\Omega (pu_x v_x + pu_y v_y)dxdy + \int_{\Gamma_2} \sigma uvds, \quad \forall u, v \in H^1(\Omega). \tag{1.1.4}$$

In (1.1.4), fixed u, then $B(u, v)$ is a linear functional of v, while v is fixed, it is a linear functional of u. In other words, suppose $\alpha_1, \alpha_2, \beta_1, \beta_2$ are arbitrary constants, then

$$\begin{aligned} B(\alpha_1 u_1 + \alpha_2 u_2, \beta_1 v_1 + \beta_2 v_2) =& \alpha_1\beta_1 B(u_1, v_1) + \alpha_1\beta_2 B(u_1, v_2) \\ &+ \alpha_2\beta_1 B(u_2, v_1) + \alpha_2\beta_2 B(u_2, v_2), \quad \forall u_1, u_2, v_1, v_2 \in H^1(\Omega). \end{aligned}$$

It is clear that (1.1.4) satisfies

(1) Symmetry,

$$B(u, v) = B(v, u). \tag{1.1.5}$$

(2) The continuity in $V \times V$, i.e., there exists a constant $M > 0$, such that

$$|B(u, v)| \leqslant M\|u\|_{1,\Omega}\|v\|_{1,\Omega}, \quad \forall u, v \in V. \tag{1.1.6}$$

(3) Coerciveness in V, i.e., there exists constant $\gamma > 0$, such that

$$B(u, u) \geqslant \gamma\|u\|_{1,\Omega}^2, \quad \forall u \in V. \tag{1.1.7}$$

Of course,

$$f(v) = \iint_\Omega fv\, dxdy + \int_{\Gamma_2} gvds$$

is a continuous linear functional in v.

The Galerkin Variational Formulation for (1.1.1): Find $u \in V$, such that

$$B(u, v) = f(v), \quad \forall v \in V. \tag{1.1.8}$$

A solution u satisfying (1.1.8) is called a *weak solution* of (1.1.1). The space V is called *admissible space* or *trial space.* On the other hand, (1.1.8) must be satisfied for every $v \in V$, therefore, V is called *test function space.* If trial and test space for the variational problem are the same Hilbert V, in this case, V is called *energy space.*

Owing to the boundary condition on Γ_2 is contained in the variational problem (1.1.8), the boundary condition on Γ_2 is called *nature boundary condition,* while the boundary condition on Γ_1 is called *essential boundary condition.*

The following proposition gives the relationship between classical solution and weak solution of (1.1.1).

Proposition 1.1 Suppose $u \in C^2(\Omega)$. If u is a classical solution of (1.1.1), then, u is the weak solution of (1.1.1). Otherwise, if u is a weak solution of (1.1.1), then u is a *classical solution* of (1.1.1).

Proof Assume that $u \in C^2(\Omega)$ is a classical solution of (1.1.1), $\forall v \in V$, multiplying both sides of (1.1.1) by v and integrating

$$-\iint_\Omega v\text{div}\,(p\nabla u)dxdy = \iint_\Omega fvdxdy,$$

which can be rewritten as

$$-\iint_\Omega (\text{div}(pv\nabla u) - p\nabla v\nabla u)dxdy = \iint_\Omega fvdxdy.$$

Applying Gauss theorem

$$-\iint_\Omega p\nabla u\nabla vdxdy - \oint_{\partial\Omega} pv\frac{\partial u}{\partial n}ds = \iint_\Omega fvdxdy.$$

In view of $v \in V$, and u satisfying boundary condition we have

$$B(u,v) = f(v), \quad \forall v \in V,$$

i.e., u satisfies (1.1.7).

Conversely, let $u \in C^2(\Omega)$ be a solution of (1.1.8), owing to $v|_{\Gamma_1} = 0$, we obtain

$$\begin{aligned}\iint_\Omega p\nabla u \nabla v dxdy &= \iint_\Omega [\mathrm{div}(pv\nabla u) - v\ \mathrm{div}(p\nabla u)]dxdy \\ &= \oint_{\partial\Omega} pv\frac{\partial u}{\partial n}ds - \iint_\Omega v\ \mathrm{div}(p\nabla u)dxdy \\ &= \int_{\Gamma_2} pv\frac{\partial u}{\partial n}ds - \iint_\Omega v\ \mathrm{div}(p\nabla u)dxdy.\end{aligned}$$

Substituting above equity into (1.1.8) leads to

$$\int_{\Gamma_2} \sigma uvds + \int_{\Gamma_2} pv\frac{\partial u}{\partial n}ds - \int_\Omega v\ \mathrm{div}(p\nabla u)dxdy = \int_\Omega fvdxdy + \int_{\Gamma_2} gvds.$$

Therefore

$$\iint_\Omega v\ [\mathrm{div}(p\nabla u) + f]dxdy + \int_{\Gamma_2} v\left(g - p\frac{\partial u}{\partial n} - \sigma u\right) ds = 0.$$

By the arbitrary of $v \in V$, it yields that u is a *classical solution* of (1.1.1). The proof is complete. □

The following Lax-Milgram theorem guarantees the existence of the Garlerkin variational problem (1.2.8).

Theorem 1.1(Lax-Milgram Theorem) Let V be a Hilbert space, $B(u,v)$ is a bilinear functional in $V \times V$ and satisfies:

Symmetry $B(u,v) = B(v,u), \quad \forall u,v \in V.$ (1.1.9)

Continuity There exists a positive constant M independent of (u,v), such that

$$|B(u,v)| \leqslant M\|u\|\ \|v\|, \quad \forall u,v \in V. \tag{1.1.10}$$

Coerciveness There exists a constant $\gamma \geqslant 0$, independent of (u) such that

$$B(u,u) \geqslant \gamma\|u\|^2, \quad \forall u \in V, \tag{1.1.11}$$

where $\|\cdot\|$ is the norm of V, f linear functional in V. Then, there exists a unique solution for Galerkin variational problem:

$$\begin{cases} \text{Find}\, u \in V, \quad \text{such that} \\ B(u,v) = f(v), \quad v \in V, \end{cases} \tag{1.1.12}$$

and the following inequality hold:

$$\|u^*\| \leqslant \frac{1}{\gamma}\|f\|_*,$$

where $\|\cdot\|_*$ is the norm in the dual space V^*, i.e.,

$$\|f\|_* = \sup_{v \in V} \frac{\langle f, v \rangle}{\|v\|}. \tag{1.1.13}$$

Proof Since $B(u, v)$ satisfies the symmetry and coerciveness, a new inner product $[u, v] = B(u, v)$ in V can be defined. Moreover, the new norm is equivalent to $\|\cdot\|$:

$$\gamma\|v\|^2 \leqslant [u, v] \leqslant M\|v\|^2.$$

Hence, $f(v)$ is also a bounded linear functional with respect to the new norm. According to Riesz theorem, there exists unique element $u^* \in V$ such that

$$[u^*, v] = f(v), \quad \forall v \in V,$$

i.e.

$$B(u^*, v) = f(v), \quad v \in V.$$

This means that u^* is a solution. On the other hand,

$$\gamma\|u^*\|^2 \leqslant B(u^*, u^*) = [u^*, u^*] = f(u^*) \leqslant \|f\|_*\|u^*\|,$$

Therefore, inequality (1.1.13) is valid. This completes our proof. □

Let quadratic functional be given by

$$J(v) = \frac{1}{2}B(v, v) - f(v). \tag{1.1.14}$$

Let us consider minimum problem of $J(v)$:

$$\begin{cases} \text{Find} \quad u \in V, \quad \text{such that} \\ J(u) = \min\limits_{v \in V} J(v). \end{cases} \tag{1.1.15}$$

Problem (1.1.15) is called Ritz variational problem for (1.1.1).

Theorem 1.2 Let V be a Hilbert space. $B(u, v)$ in $V \times V$ is a bilinear form satisfying (1.1.9)~(1.1.11). f is a linear continuous functional in V, $J(v)$ quadratic functional defined by (1.1.14). Then, for two problems (1.1.15) and (1.1.12), the following conclusions are valid:

1) If there exists a solution for any problem, then these solutions are no more than one;
2) Any solution of one problem must be the solution of other problem;
3) For any solution u^* of the problem, then,

$$J(v) - J(u^*) = \frac{1}{2}B(v - u^*, v - u^*), \quad \forall v \in V. \tag{1.1.16}$$

Proof First, let us prove the uniqueness of (1.1.12). Assume that u_1, u_2 are the solutions of (1.1.12). Let $w = u_1 - u_2$. Since

$$B(u_1, v) = f(v), \quad \forall v \in V, \qquad B(u_2, v) = f(v), \quad \forall v \in V,$$