Graduate Texts in Mathematics

Paul R. Halmos

A Hilbert Space Problem Book

Second Edition

希尔伯特空间问题集

第2版

Springer

世界图书出版公司
www.wpcbj.com.cn

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Second Edition, Revised and Enlarged

Springer-Verlag
New York Berlin Heidelberg London Paris
Tokyo Hong Kong Barcelona Budapest

图书在版编目（CIP）数据

希尔伯特空间问题集：第2版：英文/（美）哈尔莫斯（Halmos，P. R.）. —北京：世界图书出版公司北京公司，2009. 4

书名原文：A Hilbert Space Problem Book 2nd ed.

ISBN 978-7-5100-0444-5

Ⅰ. 希… Ⅱ. 哈… Ⅲ. 希尔伯特空间—英文 Ⅳ. 0177. 1

中国版本图书馆 CIP 数据核字（2009）第 048015 号

书　　名：	A Hilbert Space Problem Book 2nd ed.
作　　者：	Paul R. Halmos
中 译 名：	希尔伯特空间问题集 第2版
责任编辑：	高蓉　刘慧
出 版 者：	世界图书出版公司北京公司
印 刷 者：	三河国英印务有限公司
发　　行：	世界图书出版公司北京公司（北京朝内大街 137 号　100010）
联系电话：	010-64021602，010-64015659
电子信箱：	kjb@ wpcbj. com. cn
开　　本：	24 开
印　　张：	16. 5
版　　次：	2009 年 04 月
版权登记：	图字：01-2009-0692
书　　号：	978-7-5100-0444-5/O · 659　　**定　　价：** 38. 00 元

Mathematics Subject Classifications (1991): 46-01, 00A07, 46CXX

Library of Congress Cataloging in Publication Data
Halmos, Paul R. (Paul Richard), 1916–
A Hilbert space problem book.
(Graduate texts in mathematics; 19)
Bibliography: p.
Includes index.
1. Hilbert spaces—Problems, exercises, etc.
I. Title. II. Series.
QA322.4.H34 1982 515.7'33 82-763
AACR2

ISBN 0-387-90685-1 Springer-Verlag New York Heidelberg Berlin
ISBN 3-540-90685-1 Springer-Verlag Berlin Heidelberg New York

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To J. U. M.

Preface

The only way to learn mathematics is to do mathematics. That tenet is the foundation of the do-it-yourself, Socratic, or Texas method, the method in which the teacher plays the role of an omniscient but largely uncommunicative referee between the learner and the facts. Although that method is usually and perhaps necessarily oral, this book tries to use the same method to give a written exposition of certain topics in Hilbert space theory.

The right way to read mathematics is first to read the definitions of the concepts and the statements of the theorems, and then, putting the book aside, to try to discover the appropriate proofs. If the theorems are not trivial, the attempt might fail, but it is likely to be instructive just the same. To the passive reader a routine computation and a miracle of ingenuity come with equal ease, and later, when he must depend on himself, he will find that they went as easily as they came. The active reader, who has found out what does not work, is in a much better position to understand the reason for the success of the author's method, and, later, to find answers that are not in books.

This book was written for the active reader. The first part consists of problems, frequently preceded by definitions and motivation, and sometimes followed by corollaries and historical remarks. Most of the problems are statements to be proved, but some are questions (is it?, what is?), and some are challenges (construct, determine). The second part, a very short one, consists of hints. A hint is a word, or a paragraph, usually intended to help the reader find a solution. The hint itself is not necessarily a condensed solution of the problem; it may just point to what I regard as the heart of the matter. Sometimes a problem contains a trap, and the hint may serve to chide the reader for rushing in too recklessly. The third part, the longest, consists of solutions: proofs, answers, or constructions, depending on the nature of the problem.

The problems are intended to be challenges to thought, not legal technicalities. A reader who offers solutions in the strict sense only (this is what was asked, and here is how it goes) will miss a lot of the point, and he will miss a lot of fun. Do not just answer the question, but try to think of related questions, of generalizations (what if the operator is not normal?), and of special cases (what happens in the finite-dimensional case?). What makes the assertion true? What would make it false?

Problems in life, in mathematics, and even in this book, do not necessarily arise in increasing order of depth and difficulty. It can perfectly well happen that a relatively unsophisticated fact about operators is the best tool for the solution of an elementary-sounding problem about the geometry of vectors. Do not be discouraged if the solution of an early problem borrows from the future and uses the results of a later discussion. The logical error of circular reasoning must be avoided, of course. An insistently linear view of the intricate architecture of mathematics is, however, almost as bad: it tends to conceal the beauty of the subject and to delay or even to make impossible an understanding of the full truth.

If you cannot solve a problem, and the hint did not help, the best thing to do at first is to go on to another problem. If the problem was a statement, do not hesitate to use it later; its use, or possible misuse, may throw valuable light on the solution. If, on the other hand, you solved a problem, look at the hint, and then the solution, anyway. You may find modifications, generalizations, and specializations that you did not think of. The solution may introduce some standard nomenclature, discuss some of the history of the subject, and mention some pertinent references.

The topics treated range from fairly standard textbook material to the boundary of what is known. I made an attempt to exclude dull problems with routine answers; every problem in the book puzzled me once. I did not try to achieve maximal generality in all the directions that the problems have contact with. I tried to communicate ideas and techniques and to let the reader generalize for himself.

To get maximum profit from the book the reader should know the elementary techniques and results of general topology, measure theory, and real and complex analysis. I use, with no apology and no reference, such concepts as subbase for a topology, precompact metric spaces, Lindelöf spaces, connectedness, and the convergence of nets, and such results as the metrizability of compact spaces with a countable base, and the compactness of the Cartesian product of compact spaces. (Reference: [87].) From measure theory, I use concepts such as σ-fields and L^p spaces, and results such as that L^p convergent sequences have almost everywhere convergent subsequences, and the Lebesgue dominated convergence theorem. (Reference: [61].) From real analysis I need, at least, the facts about the derivatives of absolutely continuous functions, and the Weierstrass polynomial approximation theorem. (Reference: [120].) From complex analysis I need such things as Taylor and Laurent series, subuniform convergence, and the maximum modulus principle. (Reference: [26].)

This is not an introduction to Hilbert space theory. Some knowledge of that subject is a prerequisite; at the very least, a study of the elements of Hilbert space theory should proceed concurrently with the reading of this book. Ideally the reader should know something like the first two chapters of [50].

I tried to indicate where I learned the problems and the solutions and where further information about them is available, but in many cases I could find no reference. When I ascribe a result to someone without an accompanying bracketed reference number, I am referring to an oral communication or an unpublished preprint. When I make no ascription, I am not claiming originality; more than likely the result is a folk theorem.

The notation and terminology are mostly standard and used with no explanation. As far as Hilbert space is concerned, I follow [50], except in a few small details. Thus, for instance, I now use f and g for vectors, instead of x and y (the latter are too useful for points in measure spaces and such), and, in conformity with current fashion, I use "kernel" instead of "null-space". (The triple use of the word, to denote (1) null-space, (2) the continuous analogue of a matrix, and (3) the reproducing function associated with a functional Hilbert space, is regrettable but unavoidable; it does not seem to lead to any confusion.) Incidentally kernel and range are abbreviated as ker and ran, their orthogonal complements are abbreviated as $\ker^{\perp}$ and $\operatorname{ran}^{\perp}$, dimension is abbreviated as dim, and determinant and trace are abbreviated as det and tr. Real and imaginary parts are denoted, as usual, by Re and Im. The "signum" of a complex number z, i.e., $z/|z|$ or 0 according as $z \neq 0$ or $z = 0$, is denoted by sgn z.

The zero subspace of a Hilbert space is denoted by 0, instead of the correct, pedantic $\{0\}$. (The simpler notation is obviously more convenient, and it is not a whit more illogical than the simultaneous use of the symbol "0" for a number, a function, a vector, and an operator. I cannot imagine any circumstances where it could lead to serious error. To avoid even a momentary misunderstanding, however, I write $\{0\}$ for the set of complex numbers consisting of 0 alone.) The *co-dimension* of a subspace is the dimension of its orthogonal complement (or, equivalently, the dimension of the quotient space it defines). The symbols $\bigvee$ (as a prefix) and $\vee$ (as an infix) are used to denote spans, so that if $\mathbf{M}$ is an arbitrary set of vectors, then $\bigvee \mathbf{M}$ is the smallest closed linear manifold that includes $\mathbf{M}$; if $\mathbf{M}$ and $\mathbf{N}$ are sets of vectors, then $\mathbf{M} \vee \mathbf{N}$ is the smallest closed linear manifold that includes both $\mathbf{M}$ and $\mathbf{N}$; and if $\{\mathbf{M}_j\}$ is a family of sets of vectors, then $\bigvee_j \mathbf{M}_j$ is the smallest closed linear manifold that includes each $\mathbf{M}_j$. Subspace, by the way, means closed linear manifold, and operator means bounded linear transformation.

The arrow in a symbol such as $f_n \to f$ indicates that a sequence $\{f_n\}$ tends to the limit f; the barred arrow in $x \mapsto x^2$ denotes the function φ defined by

$\varphi(x) = x^2$. (Note that barred arrows "bind" their variables, just as integrals in calculus and quantifiers in logic bind theirs. In principle equations such as $(x \mapsto x^2)(y) = y^2$ make sense.)

Since the inner product of two vectors f and g is always denoted by (f, g), another symbol is needed for their ordered pair; I use $\langle f, g \rangle$. This leads to the systematic use of the angular bracket to enclose the coordinates of a vector, as in $\langle f_0, f_1, f_2, \ldots \rangle$. In accordance with inconsistent but widely accepted practice, I use braces to denote both sets and sequences; thus $\{x\}$ is the set whose only element is x, and $\{x_n\}$ is the sequence whose n-th term is x_n, $n = 1, 2, 3, \cdots$. This could lead to confusion, but in context it does not seem to do so. For the complex conjugate of a complex number z, I use z^*. This tends to make mathematicians nervous, but it is widely used by physicists, it is in harmony with the standard notation for the adjoints of operators, and it has typographical advantages. (The image of a set M of complex numbers under the mapping $z \mapsto z^*$ is M^*; the symbol $\overline{M}$ suggests topological closure.)

Operator theory has made much progress since the first edition of this book appeared in 1967. Some of that progress is visible in the difference between the two editions. The journal literature needs time, however, to ripen, to become understood and simplified enough for expository presentation in a book of this sort, and much of it is not yet ready for that. Even in the part that is ready, I had to choose; not everything could be fitted in. I omitted beautiful and useful facts about essential spectra, the Calkin algebra, and Toeplitz and Hankel operators, and I am sorry about that. Maybe next time.

The first edition had 199 problems; this one has $199 - 9 + 60$. I hope that the number of incorrect or awkward statements and proofs is smaller in this edition. In any event, something like ten of the problems (or their solutions) were substantially revised. (Whether the actual number is 8 or 9 or 11 or 12 depends on how a "substantial" revision is defined.) The new problems have to do with several subjects; the three most frequent ones are total sets of vectors, cyclic operators, and the weak and strong operator topologies.

Since I have been teaching Hilbert space by the problem method for many years, I owe thanks for their help to more friends among students and colleagues than I could possibly name here. I am truly grateful to them all just the same. Without them this book could not exist; it is not the sort of book that could have been written in isolation from the mathematical community. My special thanks are due to Ronald Douglas, Eric Nordgren, and Carl Pearcy for the first edition, and Donald Hadwin and David Schwab for the second. Each of them read the whole manuscript (well, almost the whole manuscript) and stopped me from making many foolish mistakes.

Santa Clara University　　　　P.R.H.

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