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Sheldon Axler
Paul Bourdon
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Harmonic Function Theory

Second Edition

调和函数理论 第2版

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Preface

Harmonic functions—the solutions of Laplace's equation—play a crucial role in many areas of mathematics, physics, and engineering. But learning about them is not always easy. At times the authors have agreed with Lord Kelvin and Peter Tait, who wrote ([18], Preface)

> There can be but one opinion as to the beauty and utility of this analysis of Laplace; but the manner in which it has been hitherto presented has seemed repulsive to the ablest mathematicians, and difficult to ordinary mathematical students.

The quotation has been included mostly for the sake of amusement, but it does convey a sense of the difficulties the uninitiated sometimes encounter.

The main purpose of our text, then, is to make learning about harmonic functions easier. We start at the beginning of the subject, assuming only that our readers have a good foundation in real and complex analysis along with a knowledge of some basic results from functional analysis. The first fifteen chapters of [15], for example, provide sufficient preparation.

In several cases we simplify standard proofs. For example, we replace the usual tedious calculations showing that the Kelvin transform of a harmonic function is harmonic with some straightforward observations that we believe are more revealing. Another example is our proof of Bôcher's Theorem, which is more elementary than the classical proofs.

We also present material not usually covered in standard treatments of harmonic functions (such as [9], [11], and [19]). The section on the Schwarz Lemma and the chapter on Bergman spaces are examples. For completeness, we include some topics in analysis that frequently slip through the cracks in a beginning graduate student's curriculum, such as real-analytic functions.

We rarely attempt to trace the history of the ideas presented in this book. Thus the absence of a reference does not imply originality on our part.

For this second edition we have made several major changes. The key improvement is a new and considerably simplified treatment of spherical harmonics (Chapter 5). The book now includes a formula for the Laplacian of the Kelvin transform (Proposition 4.6). Another addition is the proof that the Dirichlet problem for the half-space with continuous boundary data is solvable (Theorem 7.11), with no growth conditions required for the boundary function. Yet another significant change is the inclusion of generalized versions of Liouville's and Bôcher's Theorems (Theorems 9.10 and 9.11), which are shown to be equivalent. We have also added many exercises and made numerous small improvements.

In addition to writing the text, the authors have developed a software package to manipulate many of the expressions that arise in harmonic function theory. Our software package, which uses many results from this book, can perform symbolic calculations that would take a prohibitive amount of time if done without a computer. For example, the Poisson integral of any polynomial can be computed exactly. Appendix B explains how readers can obtain our software package free of charge.

The roots of this book lie in a graduate course at Michigan State University taught by one of the authors and attended by the other authors along with a number of graduate students. The topic of harmonic functions was presented with the intention of moving on to different material after introducing the basic concepts. We did not move on to different material. Instead, we began to ask natural questions about harmonic functions. Lively and illuminating discussions ensued. A freewheeling approach to the course developed; answers to questions someone had raised in class or in the hallway were worked out and then presented in class (or in the hallway). Discovering mathematics in this way was a thoroughly enjoyable experience. We will consider this book a success if some of that enjoyment shines through in these pages.

Acknowledgments

Our book has been improved by our students and by readers of the first edition. We take this opportunity to thank them for catching errors and making useful suggestions.

Among the many mathematicians who have influenced our outlook on harmonic function theory, we give special thanks to Dan Luecking for helping us to better understand Bergman spaces, to Patrick Ahern who suggested the idea for the proof of Theorem 7.11, and to Elias Stein and Guido Weiss for their book [16], which contributed greatly to our knowledge of spherical harmonics.

We are grateful to Carrie Heeter for using her expertise to make old photographs look good.

At our publisher Springer we thank the mathematics editors Thomas von Foerster (first edition) and Ina Lindemann (second edition) for their support and encouragement, as well as Fred Bartlett for his valuable assistance with electronic production.

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CHAPTER 1

Basic Properties of Harmonic Functions

Definitions and Examples

Harmonic functions, for us, live on open subsets of real Euclidean spaces. Throughout this book, n will denote a fixed positive integer greater than 1 and Ω will denote an open, nonempty subset of $\mathbf{R}^n$. A twice continuously differentiable, complex-valued function u defined on Ω is *harmonic* on Ω if

$$\Delta u \equiv 0,$$

where $\Delta = D_1{}^2 + \cdots + D_n{}^2$ and $D_j{}^2$ denotes the second partial derivative with respect to the j^{th} coordinate variable. The operator Δ is called the *Laplacian*, and the equation $\Delta u \equiv 0$ is called *Laplace's equation*. We say that a function u defined on a (not necessarily open) set $E \subset \mathbf{R}^n$ is harmonic on E if u can be extended to a function harmonic on an open set containing E.

We let $x = (x_1, \ldots, x_n)$ denote a typical point in $\mathbf{R}^n$ and let $|x| = (x_1{}^2 + \cdots + x_n{}^2)^{1/2}$ denote the Euclidean norm of x.

The simplest nonconstant harmonic functions are the coordinate functions; for example, $u(x) = x_1$. A slightly more complex example is the function on $\mathbf{R}^3$ defined by

$$u(x) = x_1{}^2 + x_2{}^2 - 2x_3{}^2 + ix_2.$$

As we will see later, the function

$$u(x) = |x|^{2-n}$$

is vital to harmonic function theory when $n > 2$; the reader should verify that this function is harmonic on $\mathbf{R}^n \setminus \{0\}$.

We can obtain additional examples of harmonic functions by differentiation, noting that for smooth functions the Laplacian commutes with any partial derivative. In particular, differentiating the last example with respect to x_1 shows that $x_1|x|^{-n}$ is harmonic on $\mathbf{R}^n \setminus \{0\}$ when $n > 2$. (We will soon prove that every harmonic function is infinitely differentiable; thus every partial derivative of a harmonic function is harmonic.)

The function $x_1|x|^{-n}$ is harmonic on $\mathbf{R}^n \setminus \{0\}$ even when $n = 2$. This can be verified directly or by noting that $x_1|x|^{-2}$ is a partial derivative of $\log|x|$, a harmonic function on $\mathbf{R}^2 \setminus \{0\}$. The function $\log|x|$ plays the same role when $n = 2$ that $|x|^{2-n}$ plays when $n > 2$. Notice that $\lim_{x\to\infty} \log|x| = \infty$, but $\lim_{x\to\infty} |x|^{2-n} = 0$; note also that $\log|x|$ is neither bounded above nor below, but $|x|^{2-n}$ is always positive. These facts hint at the contrast between harmonic function theory in the plane and in higher dimensions. Another key difference arises from the close connection between holomorphic and harmonic functions in the plane—a real-valued function on $\Omega \subset \mathbf{R}^2$ is harmonic if and only if it is locally the real part of a holomorphic function. No comparable result exists in higher dimensions.

Invariance Properties

Throughout this book, all functions are assumed to be complex valued unless stated otherwise. For k a positive integer, let $C^k(\Omega)$ denote the set of k times continuously differentiable functions on Ω; $C^\infty(\Omega)$ is the set of functions that belong to $C^k(\Omega)$ for every k. For $E \subset \mathbf{R}^n$, we let $C(E)$ denote the set of continuous functions on E.

Because the Laplacian is linear on $C^2(\Omega)$, sums and scalar multiples of harmonic functions are harmonic.

For $y \in \mathbf{R}^n$ and u a function on Ω, the *y-translate* of u is the function on $\Omega + y$ whose value at x is $u(x - y)$. Clearly, translations of harmonic functions are harmonic.

For a positive number r and u a function on Ω, the *r-dilate of u*, denoted u_r, is the function

$$(u_r)(x) = u(rx)$$

defined for x in $(1/r)\Omega = \{(1/r)w : w \in \Omega\}$. If $u \in C^2(\Omega)$, then a simple computation shows that $\Delta(u_r) = r^2(\Delta u)_r$ on $(1/r)\Omega$. Hence dilates of harmonic functions are harmonic.

Note the formal similarity between the Laplacian $\Delta = D_1{}^2 + \cdots + D_n{}^2$ and the function $|x|^2 = x_1{}^2 + \cdots + x_n{}^2$, whose level sets are spheres centered at the origin. The connection between harmonic functions and spheres is central to harmonic function theory. The mean-value property, which we discuss in the next section, best illustrates this connection. Another connection involves linear transformations on $\mathbf{R}^n$ that preserve the unit sphere; such transformations are called *orthogonal*. A linear map $T: \mathbf{R}^n \to \mathbf{R}^n$ is orthogonal if and only if $|Tx| = |x|$ for all $x \in \mathbf{R}^n$. Simple linear algebra shows that T is orthogonal if and only if the column vectors of the matrix of T (with respect to the standard basis of $\mathbf{R}^n$) form an orthonormal set.

We now show that the Laplacian commutes with orthogonal transformations; more precisely, if T is orthogonal and $u \in C^2(\Omega)$, then

$$\Delta(u \circ T) = (\Delta u) \circ T$$

on $T^{-1}(\Omega)$. To prove this, let $[t_{jk}]$ denote the matrix of T relative to the standard basis of $\mathbf{R}^n$. Then

$$D_m(u \circ T) = \sum_{j=1}^{n} t_{jm}(D_j u) \circ T,$$

where D_m denotes the partial derivative with respect to the m^{th} coordinate variable. Differentiating once more and summing over m yields

$$\begin{aligned}
\Delta(u \circ T) &= \sum_{m=1}^{n} \sum_{j,k=1}^{n} t_{km} t_{jm} (D_k D_j u) \circ T \\
&= \sum_{j,k=1}^{n} \Big(\sum_{m=1}^{n} t_{km} t_{jm} \Big) (D_k D_j u) \circ T \\
&= \sum_{j=1}^{n} (D_j D_j u) \circ T \\
&= (\Delta u) \circ T,
\end{aligned}$$

as desired. The function $u \circ T$ is called a *rotation* of u. The preceding calculation shows that rotations of harmonic functions are harmonic.

The Mean-Value Property

Many basic properties of harmonic functions follow from Green's identity (which we will need mainly in the special case when Ω is a ball):

1.1
$$\int_{\Omega}(u\Delta v - v\Delta u)\,dV = \int_{\partial\Omega}(uD_{\mathbf{n}}v - vD_{\mathbf{n}}u)\,ds.$$

Here Ω is a bounded open subset of $\mathbf{R}^n$ with smooth boundary, and u and v are C^2-functions on a neighborhood of $\overline{\Omega}$, the closure of Ω. The measure $V = V_n$ is Lebesgue volume measure on $\mathbf{R}^n$, and s denotes surface-area measure on $\partial\Omega$ (see Appendix A for a discussion of integration over balls and spheres). The symbol $D_{\mathbf{n}}$ denotes differentiation with respect to the outward unit normal $\mathbf{n}$. Thus for $\zeta \in \partial\Omega$, $(D_{\mathbf{n}}u)(\zeta) = (\nabla u)(\zeta) \cdot \mathbf{n}(\zeta)$, where $\nabla u = (D_1 u, \dots, D_n u)$ denotes the gradient of u and $\cdot$ denotes the usual Euclidean inner product.

Green's identity (1.1) follows easily from the familiar divergence theorem of advanced calculus:

1.2
$$\int_{\Omega} \operatorname{div} \mathbf{w}\, dV = \int_{\partial\Omega} \mathbf{w} \cdot \mathbf{n}\, ds.$$

Here $\mathbf{w} = (w_1, \dots, w_n)$ is a smooth vector field (a $\mathbf{C}^n$-valued function whose components are continuously differentiable) on a neighborhood of $\overline{\Omega}$, and $\operatorname{div} \mathbf{w}$, the divergence of $\mathbf{w}$, is defined to be $D_1 w_1 + \cdots + D_n w_n$. To obtain Green's identity from the divergence theorem, simply let $\mathbf{w} = u\nabla v - v\nabla u$ and compute.

The following useful form of Green's identity occurs when u is harmonic and $v \equiv 1$:

1.3
$$\int_{\partial\Omega} D_{\mathbf{n}}u\, ds = 0.$$

Green's identity is the key to the proof of the mean-value property. Before stating the mean-value property, we introduce some notation: $B(a,r) = \{x \in \mathbf{R}^n : |x - a| < r\}$ is the open ball centered at a of radius r; its closure is the closed ball $\overline{B}(a,r)$; the unit ball $B(0,1)$ is denoted by B and its closure by $\overline{B}$. When the dimension is important we write B_n in place of B. The unit sphere, the boundary of B, is denoted by S; normalized surface-area measure on S is denoted by σ (so that $\sigma(S) = 1$). The measure σ is the unique Borel probability measure on S that is rotation invariant (meaning $\sigma(T(E)) = \sigma(E)$ for every Borel set $E \subset S$ and every orthogonal transformation T).

1.4 Mean-Value Property: *If u is harmonic on $\overline{B}(a,r)$, then u equals the average of u over $\partial B(a,r)$. More precisely,*

$$u(a) = \int_S u(a + r\zeta)\, d\sigma(\zeta).$$

PROOF: First assume that $n > 2$. Without loss of generality we may assume that $B(a,r) = B$. Fix $\varepsilon \in (0,1)$. Apply Green's identity (1.1) with $\Omega = \{x \in \mathbf{R}^n : \varepsilon < |x| < 1\}$ and $v(x) = |x|^{2-n}$ to obtain

$$0 = (2-n)\int_S u\, ds - (2-n)\varepsilon^{1-n}\int_{\varepsilon S} u\, ds - \int_S D_{\mathbf{n}} u\, ds - \varepsilon^{2-n}\int_{\varepsilon S} D_{\mathbf{n}} u\, ds.$$

By 1.3, the last two terms are 0, thus

$$\int_S u\, ds = \varepsilon^{1-n}\int_{\varepsilon S} u\, ds,$$

which is the same as

$$\int_S u\, d\sigma = \int_S u(\varepsilon\zeta)\, d\sigma(\zeta).$$

Letting $\varepsilon \to 0$ and using the continuity of u at 0, we obtain the desired result.

The proof when $n = 2$ is the same, except that $|x|^{2-n}$ should be replaced by $\log|x|$. ∎

Harmonic functions also have a mean-value property with respect to volume measure. The polar coordinates formula for integration on $\mathbf{R}^n$ is indispensable here. The formula states that for a Borel measurable, integrable function f on $\mathbf{R}^n$,

1.5
$$\frac{1}{nV(B)}\int_{\mathbf{R}^n} f\,dV = \int_0^\infty r^{n-1}\int_S f(r\zeta)\,d\sigma(\zeta)\,dr$$

(see [15], Chapter 8, Exercise 6). The constant $nV(B)$ arises from the normalization of σ (choosing f to be the characteristic function of B shows that $nV(B)$ is the correct constant).

1.6 Mean-Value Property, Volume Version: *If u is harmonic on $\overline{B}(a,r)$, then $u(a)$ equals the average of u over $B(a,r)$. More precisely,*

$$u(a) = \frac{1}{V(B(a,r))}\int_{B(a,r)} u\,dV.$$

PROOF: We can assume that $B(a,r) = B$. Apply the polar coordinates formula (1.5) with f equal to u times the characteristic function of B, and then use the spherical mean-value property (Theorem 1.4). ■

We will see later (1.24 and 1.25) that the mean-value property characterizes harmonic functions.

We conclude this section with an application of the mean value property. We have seen that a real-valued harmonic function may have an isolated (nonremovable) singularity; for example, $|x|^{2-n}$ has an isolated singularity at 0 if $n > 2$. However, a real-valued harmonic function u cannot have isolated zeros.

1.7 Corollary: *The zeros of a real-valued harmonic function are never isolated.*

PROOF: Suppose u is harmonic and real valued on Ω, $a \in \Omega$, and $u(a) = 0$. Let $r > 0$ be such that $\overline{B}(a,r) \subset \Omega$. Because the average of u over $\partial B(a,r)$ equals 0, either u is identically 0 on $\partial B(a,r)$ or u takes on both positive and negative values on $\partial B(a,r)$. In the later case, the connectedness of $\partial B(a,r)$ implies that u has a zero on $\partial B(a,r)$.

Thus u has a zero on the boundary of every sufficiently small ball centered at a, proving that a is not an isolated zero of u. ■

The hypothesis that u is real valued is needed in the preceding corollary. This is no surprise when $n = 2$, because nonconstant holomorphic functions have isolated zeros. When $n \geq 2$, the harmonic function