An Introduction

Theory of Spinors

旋量理论导论

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TO OUR GRANDCHILDREN

Nadav, Guy, Daniel, Shelly

and

Michaela

Preface

This is a textbook intended for advanced undergraduate and graduate students in physics and mathematics, as well as a reference for researchers. The book is based on lectures given during the years at the Ben Gurion University, Israel. Spinors are used extensively in physics; it is widely accepted that they are more fundamental than tensors and the easy way to see this fact is the results obtained in general relativity theory by using spinors, results that could not have been obtained by using tensor methods only. The book is written for the general physicist and not only to the workers in general relativity, even though the latter will find it most useful since it includes all what is needed in that theory.

But the foundations of the concept of spinors are groups; spinors appear as representations of groups. In this text we give a wide exposition to the relationship between the spinors and the representations of the groups. As is well known, both the spinors and the representations are widely used in the theory of elementary particles.

After presenting the origin of spinors from representation theory we, nevertheless, apply the theory of spinors to general relativity theory, and a part of the book is devoted to curved spacetime applications.

In the first four chapters we present the group-theoretical foundations of the concept of two-component spinors. Chapter 1 starts with an introduction to group theory emphasizing the rotation group. This followed by discussing representation theory in Chapter 2, including a brief outline of the infinite-dimensional case. Chapters 3 and 4 discuss in detail the Lorentz and the SL(2,C) groups. Here we give an extensive discussion on how two-component spinors emerge from the finite-dimensional representations of the group SL(2,C). Chapter 4 also includes the derivation of infinite-dimensional spinors as a generalization to the two-component spinors.

In Chapters 5 and 6 we apply the two-component spinors to a variety of problems in curved spacetime. In Chapter 5 we discuss the Maxwell, Dirac and Pauli spinors. Also given in this chapter the passage to the curved spacetime of spinors. The gravitational field spinors are subsequently discussed in detail in Chapter 6. Here we derive the curvature spinor and give the spinors equivalent to the Riemann, Weyl, Ricci and Einstein tensors.

In Chapter 7 we present the gauge field spinors and discuss their geometrical properties. As is well known, gauge fields are extremely important nowadays. The Euclidean gauge field spinors are finally discussed in Chapter 8.

All chapters of the book start with the ordinary physical material before introducing the spinors of that subject. Thus, for instance, the chapters dealing with the Lorentz group and gravitation start with detailed discussion of the theories of special relativity and general relativity.

It is a pleasure to thank our wifes Elisheva and Tova for creating the necessary atmosphere and for their patience while writing this book. We are grateful to the many students who attended the courses in spinors during the years for their suggestions which led to a better presentation of the material in the book. We also want to thank Silvia Behar for her help with the Index of the book. Finally, we want to thank Julia Goldbaum for the excellent job of typing the book, prepairing the Index, and for the many suggestions for improvements.

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Contents

Chapter 1

Introduction to Group Theory

In this chapter a brief discussion on group theory is given. This includes the concept of group and subgroup, normal subgroup and factor group. Isomorphism and homomorphism are subsequently discussed. This then followed by introducing the rotation group and the group SU(2), the aggregate of unitary matrices of order two and determinant unity. A homomorphism between the pure rotation group and the group SU(2) is subsequently established. The chapter is concluded with presenting invariant integrals over the groups.

1.1 Review of Group Theory

In this section the fundamental concepts of group theory are briefly presented. For details the reader is refered to the books of Pontrjagin, van der Waerden and others suggested at the end of the chapter.

1.1.1 Group and Subgroup

A non-empty set G of elements a, b, $c, \ldots$, such as numbers, mappings, transformations, is called a *group* if the following *axioms* are satisfied:

(1) There exists an operation in the set G which associates to each two elements a and b of G a third element c of G. This operation is called

multiplication, and the element c is called the *product* of a and b, denoted by $c = ab$;

(2) The multiplication is *associative*, namely, if a, b and c are elements of G, then $(ab)\,c = a\,(bc)$;

(3) The set G contains a *right identity*, namely, there exists an element e such that $ae = a$ for each element a of G; and

(4) For each element a of G there exists a *right inverse* element, denoted by a^{-1}, such that $aa^{-1} = e$.

If the set G is finite, then the group G is called *finite* and the number of elements of G is called its *order*. Otherwise, the group G is called *infinite*. If the product of any two elements a and b of G is commutative, namely, $ab = ba$, the group is called *abelian*. In abelian groups the multiplication notation ab is replaced by an addition notation $a + b$, and the group operation is called *addition*. The identity is called *zero* and denoted by 0, and the inverse of a is called the *negative* of a and denoted by $-a$.

Since the product of group elements is associative, one writes for $(ab)\,c = a\,(bc)$ simply abc and for $(a + b) + c = a + (b + c)$ just $a + b + c$. The same holds for products of any number of elements. One can easily show (see Problem 1.1) that a right identity e is also a left identity, namely, $ea = a$, for any element a of G.

Likewise, a right inverse a^{-1} of a is also a left inverse, $a^{-1}a = e$. Hence the inverse of a^{-1} is simply a. Moreover, it follows that both the identity and the inverse are unique. This allows the use of the notation of the notation of algebra such as $a^{m+1} = a^m a$, with $a^1 = a$, for any natural number m. Negative powers of a are introduced by $a^{-m} = \left(a^{-1}\right)^m$, $a^0 = e$. Hence $a^p a^q = a^{p+q}$, and $(a^p)^q = a^{pq}$, where p and q are integers.

An example of a group is the set of all nonzero rational numbers, if the rule of combination is ordinary multiplication. The identity is the number 1.

Another example of a group whose elements are not numbers is the aggregate of rotations of a plane or of space about a fixed point. Two rotations a and b are combined by performing the rotations successively. If b is carried out first and then a, the same result, i.e. the same final position of all points of the space, may also be obtained by a single rotation, denoted by ab. The group of rotations in space is an example of non-abelian group since it is not immaterial whether one performs first the rotation a and then b, or first b and then a. The identity of the rotation group is the identical transformation that leaves every point in its original position. The inverse of a rotation is the rotation in the opposite sense which cancels the first one.

A set H of elements of a group G is called a *subgroup* of G if it is a group with the same law of multiplication which operates in G. A necessary and sufficient condition for a subset H of a group G to be a subgroup is that if H contains two elements a and b it must also contain the element ab^{-1} (see Problem 1.2).

1.1.2 Normal Subgroup and Factor Group

Let G be a group and H a subgroup, and let a and b be two elements of G. One calles a and b *equivalent*, $a \propto b$, if ab^{-1} is an element of H. The group G is thus devided into classes of equivalent elements each called a *right coset* of H relative to G. It follows that if A is a right coset of H and a is an element of A then $A = Ha$. Moreover, every set of the form Hb is a right coset and the subgroup H itself is one of the cosets. One can also introduce *left cosets* of H, written in the form aH. They are obtained from an equivalence relation such that $a \propto b$ if $a^{-1}b$ belongs to H.

A subgroup N of a group G is called an *invariant* or *normal subgroup* of G if for every element n of N and a of G the element $a^{-1}na$ belongs to N. It follows that a necessary and sufficient condition for right and left cosets of a subgroup N to coincide is that N be a normal subgroup. Every group has at least two normal subgroups, the subgroup which includes only the identity, and the subgroup which coincides with the group itself. A group which has no normal subgroup except for these two subgroups is called *simple*.

If N is a normal subgroup of a group G and A and B are two cosets of N, $A = Na$, $B = Nb$, then AB is also a coset of N. The multiplication of cosets thus defined satisfies the group axioms, and the set of all cosets is called the *factor group* of G by the normal subgroup N and is denoted by G/N.

1.1.3 Isomorphism and Homomorphism

A mapping f of a group G on another group G' is called *isomorphism* if it (1) is one-to-one; and (2) preserves the multiplication. G and G' are then called *isomorphic*. The inverse f^{-1} of an isomorphism f is itself an isomorphism. An isomorphism of a group onto itself is called *automorphism*. The aggregate of all automorphisms of a group forms a group.

A mapping f of a group G on another group G' is called *homomorphism* if it preserves the operation of multiplication. The set N of all elements of G which go over into the identity of G' under the homomorphism is called the *kernel* of the homomorphism. If the kernel coincides with the identity of G then the homomorphism is an isomorphism. It follows that N is a normal of G, and G' is isomorphic to G/N. The isomorphism between G' and G/N is called the *natural isomorphism.*

The mapping f of a group G on G/N defined by associating with each element a of G the element $f(a) = A$ of G/N containing a is a homomorphism, called the *natural homomorphism* of a group on its factor group. If f is a homomorphism of a group G on another group G' and H is a (normal) subgroup of G, then $f(H)$ is a (normal) subgroup of G'. If f is a homomorphism of a group G on another group G', and g is a homomorphism of G' on a third group G'', then the mapping gf is a homomorphism of G on G''.

One finally notes that if f is a homomorphism of a group G on part of another group G' then the set of all elements of G' which are images of elements of G forms a subgroup of G'. Also, if $f^{-1}(H')$ is the set of all elements of G which go into $H' \subset G'$ under the homomorphism f, and if H' is a (normal) subgroup of the group G', then $f^{-1}(H')$ is also a (normal) subgroup of the group G.

1.2 The Pure Rotation Group SO(3)

A linear transformation g of the variables x_1, x_2, and x_3, which leaves the form $x_1^2 + x_2^2 + x_3^2$ invariant, is called a *three-dimensional rotation.* The aggregate of all such linear transformations g forms a continuous group, which is isomorphic to the set of all *real orthogonal* (namely, $gg^t = 1$, where g^t is the transposed of g) 3-dimensional matrices and is known as the *three-dimensional rotation group.* One can easily show that the determinant of every orthogonal matrix is equal to either $+1$, in which case the transformation describes *pure rotation*, or to -1, in which case it describes a *rotation-reflection.* The aggregate of all pure rotations forms a group, which is a subgroup of the 3-dimensional rotation group, and is known as the *pure rotation group.* We will be concerned with the 3-dimensional pure rotation group. This group is denoted by us by SO(3). (For more details, see in the sequel.)

1.2.1 The Euler Angles

Let g be an element of the group SO(3), i.e., a 3-dimensional orthogonal matrix with determinant unity. It is well known that one then can express each such element in terms of a set of three parameters. An example of