

# Graduate Texts in Mathematics

Brian C. Hall

## Lie Groups, Lie Algebras, and Representations

An Elementary  
Introduction

李群、李代数和表示论

Springer

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With 31 Illustrations

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## Preface

This book provides an introduction to Lie groups, Lie algebras, and representation theory, aimed at graduate students in mathematics and physics. Although there are already several excellent books that cover many of the same topics, this book has two distinctive features that I hope will make it a useful addition to the literature. First, it treats Lie groups (not just Lie algebras) in a way that minimizes the amount of manifold theory needed. Thus, I neither assume a prior course on differentiable manifolds nor provide a condensed such course in the beginning chapters. Second, this book provides a gentle introduction to the machinery of semisimple groups and Lie algebras by treating the representation theory of  $SU(2)$  and  $SU(3)$  in detail before going to the general case. This allows the reader to see roots, weights, and the Weyl group “in action” in simple cases before confronting the general theory.

The standard books on Lie theory begin immediately with the general case: a smooth manifold that is also a group. The Lie algebra is then defined as the space of left-invariant vector fields and the exponential mapping is defined in terms of the flow along such vector fields. This approach is undoubtedly the right one in the long run, but it is rather abstract for a reader encountering such things for the first time. Furthermore, with this approach, one must either assume the reader is familiar with the theory of differentiable manifolds (which rules out a substantial part of one's audience) or one must spend considerable time at the beginning of the book explaining this theory (in which case, it takes a long time to get to Lie theory proper).

My way out of this dilemma is to consider only matrix groups (i.e., closed subgroups of  $GL(n; \mathbb{C})$ ). (Others before me have taken such an approach, as discussed later.) Every such group is a Lie group, and although not every Lie group is of this form, most of the interesting examples are. The exponential of a matrix is then defined by the usual power series, and the Lie algebra  $\mathfrak{g}$  of a closed subgroup  $G$  of  $GL(n; \mathbb{C})$  is defined to be the set of matrices  $X$  such that  $\exp(tX)$  lies in  $G$  for all real numbers  $t$ . One can show that  $\mathfrak{g}$  is, indeed, a Lie algebra (i.e., a vector space and closed under commutators). The usual elementary results can all be proved from this point of view: the image of the

exponential mapping contains a neighborhood of the identity; in a connected group, every element is a product of exponentials; every continuous group homomorphism induces a Lie algebra homomorphism. (These results show that every matrix group is a smooth embedded submanifold of  $GL(n; \mathbb{C})$ , and hence a Lie group.)

I also address two deeper results: that in the simply-connected case, every Lie algebra homomorphism induces a group homomorphism and that there is a one-to-one correspondence between subalgebras  $\mathfrak{h}$  of  $\mathfrak{g}$  and connected Lie subgroups  $H$  of  $G$ . The usual approach to these theorems makes use of the Frobenius theorem. Although this is a fundamental result in analysis, it is not easily stated (let alone proved) and it is not especially Lie-theoretic. My approach is to use, instead, the Baker–Campbell–Hausdorff theorem. This theorem is more elementary than the Frobenius theorem and arguably gives more intuition as to why the above-mentioned results are true. I begin with the technically simpler case of the Heisenberg group (where the Baker–Campbell–Hausdorff series terminates after the first commutator term) and then proceed to the general case.

Appendix C gives two examples of Lie groups that are not matrix Lie groups. Both examples are constructed from matrix Lie groups: One is the universal cover of  $SL(n; \mathbb{R})$  and the other is the quotient of the Heisenberg group by a discrete central subgroup. These examples show the limitations of working with matrix Lie groups, namely that important operations such as the of taking quotients and covers do not preserves the class of matrix Lie groups. In the long run, then, the theory of matrix Lie groups is not an acceptable substitute for general Lie group theory. Nevertheless, I feel that the matrix approach is suitable for a first course in the subject not only because most of the interesting examples of Lie groups are matrix groups but also because all of the theorems I will discuss for the matrix case continue to hold for general Lie groups. In fact, most of the proofs are the same in the general case, *except* that in the general case, one needs to spend a lot more time setting up the basic notions before one can begin.

In addressing the theory of semisimple groups and Lie algebras, I use representation theory as a motivation for the structure theory. In particular, I work out in detail the representation theory of  $SU(2)$  (or, equivalently,  $\mathfrak{sl}(2; \mathbb{C})$ ) and  $SU(3)$  (or, equivalently,  $\mathfrak{sl}(3; \mathbb{C})$ ) before turning to the general semisimple case. The  $\mathfrak{sl}(3; \mathbb{C})$  case (more so than just the  $\mathfrak{sl}(2; \mathbb{C})$  case) illustrates in a concrete way the significance of the Cartan subalgebra, the roots, the weights, and the Weyl group. In the general semisimple case, I keep the representation theory at the fore, introducing at first only as much structure as needed to state the theorem of the highest weight. I then turn to a more detailed look at root systems, including two- and three-dimensional examples, Dynkin diagrams, and a discussion (without proof) of the classification. This portion of the text includes numerous images of the relevant structures (root systems, lattices of dominant integral elements, and weight diagrams) in ranks two and three.

I take full advantage, in treating the semisimple theory, of the correspondence established earlier between the representations of a simply-connected group and the representations of its Lie algebra. So, although I treat things from the point of view of complex semisimple Lie algebras, I take advantage of the characterization of such algebras as ones isomorphic to the complexification of the Lie algebra of a compact simply-connected Lie group  $K$ . (Although, for the purposes of this book, we could take this as the definition of a complex semisimple Lie algebra, it is equivalent to the usual algebraic definition.) Having the compact group at our disposal simplifies several issues. First and foremost, it implies the complete reducibility of the representations. Second, it gives a simple construction of Cartan subalgebras, as the complexification of any maximal abelian subalgebra of the Lie algebra of  $K$ . Third, it gives a more transparent construction of the Weyl group, as  $W = N(T)/T$ , where  $T$  is a maximal torus in  $K$ . This description makes it evident, for example, why the weights of any representation are invariant under the action of  $W$ . Thus, my treatment is a mixture of the Lie algebra approach of Humphreys (1972) and the compact group approach of Bröcker and tom Dieck (1985) or Simon (1996).

This book is intended to supplement rather than replace the standard texts on Lie theory. I recommend especially four texts for further reading: the book of Lee (2003) for manifold theory and the relationship between Lie groups and Lie algebras, the book of Humphreys (1972) for the Lie algebra approach to representation theory, the book of Bröcker and tom Dieck (1985) for the compact-group approach to representation theory, and the book of Fulton and Harris (1991) for numerous examples of representations of the classical groups. There are, of course, many other books worth consulting; some of these are listed in the Bibliography.

I hope that by keeping the mathematical prerequisites to a minimum, I have made this book accessible to students in physics as well as mathematics. Although much of the material in the book is widely used in physics, physics students are often expected to pick up the material by osmosis. I hope that they can benefit from a treatment that is elementary but systematic and mathematically precise. In Appendix A, I provide a quick introduction to the theory of groups (not necessarily Lie groups), which is not as standard a part of the physics curriculum as it is of the mathematics curriculum.

The main prerequisite for this book is a solid grounding in linear algebra, especially eigenvectors and the notion of diagonalizability. A quick review of the relevant material is provided in Appendix B. In addition to linear algebra, only elementary analysis is needed: limits, derivatives, and an occasional use of compactness and the inverse function theorem.

There are, to my knowledge, five other treatments of Lie theory from the matrix group point of view. These are (in order of publication) the book *Linear Lie Groups*, by Hans Freudenthal and H. de Vries, the book *Matrix Groups*, by Morton L. Curtis, the article "Very Basic Lie Theory," by Roger Howe, and the recent books *Matrix Groups: An Introduction to Lie Group Theory*,

by Andrew Baker, and *Lie Groups: An Introduction Through Linear Groups*, by Wulf Rossmann. (All of these are listed in the Bibliography.) The book of Freudenthal and de Vries covers a lot of ground, but its unorthodox style and notation make it rather inaccessible. The works of Curtis, Howe, and Baker overlap considerably, in style and content, with the first two chapters of this book, but do not attempt to cover as much ground. For example, none of them treats representation theory or the Baker–Campbell–Hausdorff formula. The book of Rossmann has many similarities with this book, including the use of the Baker–Campbell–Hausdorff formula. However, Rossmann’s book is a bit different at the technical level, in that he considers arbitrary subgroups of  $GL(n; \mathbb{C})$ , with no restriction on the topology.

Although the organization of this book is, I believe, substantially different from that of other books on the subject, I make no claim to originality in any of the proofs. I myself learned most of the material here from books listed in the Bibliography, especially Humphreys (1972), Bröcker and tom Dieck (1985), and Miller (1972).

I am grateful to many who made corrections, large and small, to the text before publication, including Ed Bueler, Wesley Calvert, Tom Goebeler, Ruth Gornet, Keith Hubbard, Wicharn Lewkeeratiyutkul, Jeffrey Mitchell, Ambar Sengupta, and Erdinch Tatar. I am grateful as well to those who have pointed out errors in the first printing (which have been corrected in this, the second printing), including Moshe Adrian, Kamthorn Chailuek, Paul Gibson, Keith Hubbard, Dennis Muhonen, Jason Quinn, Rebecca Weber, and Reed Wickner.

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I welcome comments by e-mail at [bhall@nd.edu](mailto:bhall@nd.edu). Please visit my web site at <http://www.nd.edu/~bhall/> for more information, including an up-to-date list of corrections and many more color pictures than could be included in the book.

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(continued after index)

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# Contents

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## Part I General Theory

---

<b>1</b>	<b>Matrix Lie Groups</b> .....	<b>3</b>
1.1	Definition of a Matrix Lie Group .....	3
1.1.1	Counterexamples .....	4
1.2	Examples of Matrix Lie Groups .....	4
1.2.1	The general linear groups $GL(n; \mathbb{R})$ and $GL(n; \mathbb{C})$ .....	4
1.2.2	The special linear groups $SL(n; \mathbb{R})$ and $SL(n; \mathbb{C})$ .....	5
1.2.3	The orthogonal and special orthogonal groups, $O(n)$ and $SO(n)$ .....	5
1.2.4	The unitary and special unitary groups, $U(n)$ and $SU(n)$ .....	6
1.2.5	The complex orthogonal groups, $O(n; \mathbb{C})$ and $SO(n; \mathbb{C})$ ..	6
1.2.6	The generalized orthogonal and Lorentz groups .....	7
1.2.7	The symplectic groups $Sp(n; \mathbb{R})$ , $Sp(n; \mathbb{C})$ , and $Sp(n)$ ...	7
1.2.8	The Heisenberg group $H$ .....	8
1.2.9	The groups $\mathbb{R}^*$ , $\mathbb{C}^*$ , $S^1$ , $\mathbb{R}$ , and $\mathbb{R}^n$ .....	9
1.2.10	The Euclidean and Poincaré groups $E(n)$ and $P(n; 1)$ ...	9
1.3	Compactness .....	11
1.3.1	Examples of compact groups .....	11
1.3.2	Examples of noncompact groups .....	11
1.4	Connectedness .....	12
1.5	Simple Connectedness .....	15
1.6	Homomorphisms and Isomorphisms .....	17
1.6.1	Example: $SU(2)$ and $SO(3)$ .....	18
1.7	The Polar Decomposition for $SL(n; \mathbb{R})$ and $SL(n; \mathbb{C})$ .....	19
1.8	Lie Groups .....	20
1.9	Exercises .....	23
<b>2</b>	<b>Lie Algebras and the Exponential Mapping</b> .....	<b>27</b>
2.1	The Matrix Exponential .....	27
2.2	Computing the Exponential of a Matrix .....	30

2.2.1	Case 1: $X$ is diagonalizable	30
2.2.2	Case 2: $X$ is nilpotent	31
2.2.3	Case 3: $X$ arbitrary	32
2.3	The Matrix Logarithm	32
2.4	Further Properties of the Matrix Exponential	35
2.5	The Lie Algebra of a Matrix Lie Group	38
2.5.1	Physicists' Convention	39
2.5.2	The general linear groups	39
2.5.3	The special linear groups	40
2.5.4	The unitary groups	40
2.5.5	The orthogonal groups	40
2.5.6	The generalized orthogonal groups	41
2.5.7	The symplectic groups	41
2.5.8	The Heisenberg group	41
2.5.9	The Euclidean and Poincaré groups	42
2.6	Properties of the Lie Algebra	43
2.7	The Exponential Mapping	48
2.8	Lie Algebras	53
2.8.1	Structure constants	56
2.8.2	Direct sums	56
2.9	The Complexification of a Real Lie Algebra	56
2.10	Exercises	58
<b>3</b>	<b>The Baker–Campbell–Hausdorff Formula</b>	<b>63</b>
3.1	The Baker–Campbell–Hausdorff Formula for the Heisenberg Group	63
3.2	The General Baker–Campbell–Hausdorff Formula	67
3.3	The Derivative of the Exponential Mapping	70
3.4	Proof of the Baker–Campbell–Hausdorff Formula	73
3.5	The Series Form of the Baker–Campbell–Hausdorff Formula	74
3.6	Group Versus Lie Algebra Homomorphisms	76
3.7	Covering Groups	80
3.8	Subgroups and Subalgebras	82
3.9	Exercises	88
<b>4</b>	<b>Basic Representation Theory</b>	<b>91</b>
4.1	Representations	91
4.2	Why Study Representations?	94
4.3	Examples of Representations	95
4.3.1	The standard representation	95
4.3.2	The trivial representation	96
4.3.3	The adjoint representation	96
4.3.4	Some representations of $SU(2)$	97
4.3.5	Two unitary representations of $SO(3)$	99
4.3.6	A unitary representation of the reals	100

4.3.7	The unitary representations of the Heisenberg group ...	100
4.4	The Irreducible Representations of $\mathfrak{su}(2)$ .....	101
4.5	Direct Sums of Representations .....	106
4.6	Tensor Products of Representations .....	107
4.7	Dual Representations .....	112
4.8	Schur's Lemma .....	113
4.9	Group Versus Lie Algebra Representations .....	115
4.10	Complete Reducibility .....	118
4.11	Exercises .....	121

---

## Part II Semisimple Theory

---

<b>5</b>	<b>The Representations of <math>\mathbf{SU}(3)</math></b> .....	<b>127</b>
5.1	Introduction .....	127
5.2	Weights and Roots .....	129
5.3	The Theorem of the Highest Weight .....	132
5.4	Proof of the Theorem .....	135
5.5	An Example: Highest Weight $(1, 1)$ .....	140
5.6	The Weyl Group .....	142
5.7	Weight Diagrams .....	149
5.8	Exercises .....	152
<b>6</b>	<b>Semisimple Lie Algebras</b> .....	<b>155</b>
6.1	Complete Reducibility and Semisimple Lie Algebras .....	156
6.2	Examples of Reductive and Semisimple Lie Algebras .....	161
6.3	Cartan Subalgebras .....	162
6.4	Roots and Root Spaces .....	164
6.5	Inner Products of Roots and Co-roots .....	170
6.6	The Weyl Group .....	173
6.7	Root Systems .....	180
6.8	Positive Roots .....	181
6.9	The $\mathfrak{sl}(n; \mathbb{C})$ Case .....	182
6.9.1	The Cartan subalgebra .....	182
6.9.2	The roots .....	182
6.9.3	Inner products of roots .....	183
6.9.4	The Weyl group .....	184
6.9.5	Positive roots .....	184
6.10	Uniqueness Results .....	184
6.11	Exercises .....	185
<b>7</b>	<b>Representations of Complex Semisimple Lie Algebras</b> .....	<b>191</b>
7.1	Integral and Dominant Integral Elements .....	192
7.2	The Theorem of the Highest Weight .....	194
7.3	Constructing the Representations I: Verma Modules .....	200

7.3.1	Verma modules	200
7.3.2	Irreducible quotient modules	202
7.3.3	Finite-dimensional quotient modules	204
7.3.4	The $\mathfrak{sl}(2; \mathbb{C})$ case	208
7.4	Constructing the Representations II: The Peter–Weyl Theorem	209
7.4.1	The Peter–Weyl theorem	210
7.4.2	The Weyl character formula	211
7.4.3	Constructing the representations	213
7.4.4	Analytically integral versus algebraically integral elements	215
7.4.5	The $SU(2)$ case	216
7.5	Constructing the Representations III: The Borel–Weil Construction	218
7.5.1	The complex-group approach	218
7.5.2	The setup	220
7.5.3	The strategy	222
7.5.4	The construction	225
7.5.5	The $SL(2; \mathbb{C})$ case	229
7.6	Further Results	230
7.6.1	Duality	230
7.6.2	The weights and their multiplicities	232
7.6.3	The Weyl character formula and the Weyl dimension formula	234
7.6.4	The analytical proof of the Weyl character formula	236
7.7	Exercises	240
8	More on Roots and Weights	243
8.1	Abstract Root Systems	243
8.2	Duality	248
8.3	Bases and Weyl Chambers	249
8.4	Integral and Dominant Integral Elements	254
8.5	Examples in Rank Two	256
8.5.1	The root systems	256
8.5.2	Connection with Lie algebras	257
8.5.3	The Weyl groups	257
8.5.4	Duality	258
8.5.5	Positive roots and dominant integral elements	258
8.5.6	Weight diagrams	259
8.6	Examples in Rank Three	262
8.7	Additional Properties	263
8.8	The Root Systems of the Classical Lie Algebras	265
8.8.1	The orthogonal algebras $\mathfrak{so}(2n; \mathbb{C})$	265
8.8.2	The orthogonal algebras $\mathfrak{so}(2n + 1; \mathbb{C})$	266
8.8.3	The symplectic algebras $\mathfrak{sp}(n; \mathbb{C})$	268
8.9	Dynkin Diagrams and the Classification	269

8.10	The Root Lattice and the Weight Lattice .....	273
8.11	Exercises .....	276
<b>A</b>	<b>A Quick Introduction to Groups .....</b>	<b>279</b>
A.1	Definition of a Group and Basic Properties .....	279
A.2	Examples of Groups .....	281
A.2.1	The trivial group .....	282
A.2.2	The integers .....	282
A.2.3	The reals and $\mathbb{R}^n$ .....	282
A.2.4	Nonzero real numbers under multiplication .....	282
A.2.5	Nonzero complex numbers under multiplication .....	282
A.2.6	Complex numbers of absolute value 1 under multiplication .....	283
A.2.7	The general linear groups .....	283
A.2.8	Permutation group (symmetric group) .....	283
A.2.9	Integers <b>mod</b> $n$ .....	283
A.3	Subgroups, the Center, and Direct Products .....	284
A.4	Homomorphisms and Isomorphisms .....	285
A.5	Quotient Groups .....	286
A.6	Exercises .....	289
<b>B</b>	<b>Linear Algebra Review .....</b>	<b>291</b>
B.1	Eigenvectors, Eigenvalues, and the Characteristic Polynomial .....	291
B.2	Diagonalization .....	293
B.3	Generalized Eigenvectors and the SN Decomposition .....	294
B.4	The Jordan Canonical Form .....	296
B.5	The Trace .....	296
B.6	Inner Products .....	297
B.7	Dual Spaces .....	299
B.8	Simultaneous Diagonalization .....	299
<b>C</b>	<b>More on Lie Groups .....</b>	<b>303</b>
C.1	Manifolds .....	303
C.1.1	Definition .....	303
C.1.2	Tangent space .....	304
C.1.3	Differentials of smooth mappings .....	305
C.1.4	Vector fields .....	306
C.1.5	The flow along a vector field .....	307
C.1.6	Submanifolds of vector spaces .....	308
C.1.7	Complex manifolds .....	309
C.2	Lie Groups .....	309
C.2.1	Definition .....	309
C.2.2	The Lie algebra .....	310
C.2.3	The exponential mapping .....	311
C.2.4	Homomorphisms .....	311

## XIV Contents

C.2.5	Quotient groups and covering groups.....	312
C.2.6	Matrix Lie groups as Lie groups.....	313
C.2.7	Complex Lie groups.....	313
C.3	Examples of Nonmatrix Lie Groups.....	314
C.4	Differential Forms and Haar Measure.....	318
<b>D</b>	<b>Clebsch–Gordan Theory for <math>SU(2)</math> and the Wigner–Eckart Theorem.....</b>	<b>321</b>
D.1	Tensor Products of $sl(2; \mathbb{C})$ Representations.....	321
D.2	The Wigner–Eckart Theorem.....	324
D.3	More on Vector Operators.....	328
<b>E</b>	<b>Computing Fundamental Groups of Matrix Lie Groups.....</b>	<b>331</b>
E.1	The Fundamental Group.....	331
E.2	The Universal Cover.....	332
E.3	Fundamental Groups of Compact Lie Groups I.....	333
E.4	Fundamental Groups of Compact Lie Groups II.....	336
E.5	Fundamental Groups of Noncompact Lie Groups.....	342
	<b>References.....</b>	<b>345</b>
	<b>Index.....</b>	<b>347</b>



**General Theory**