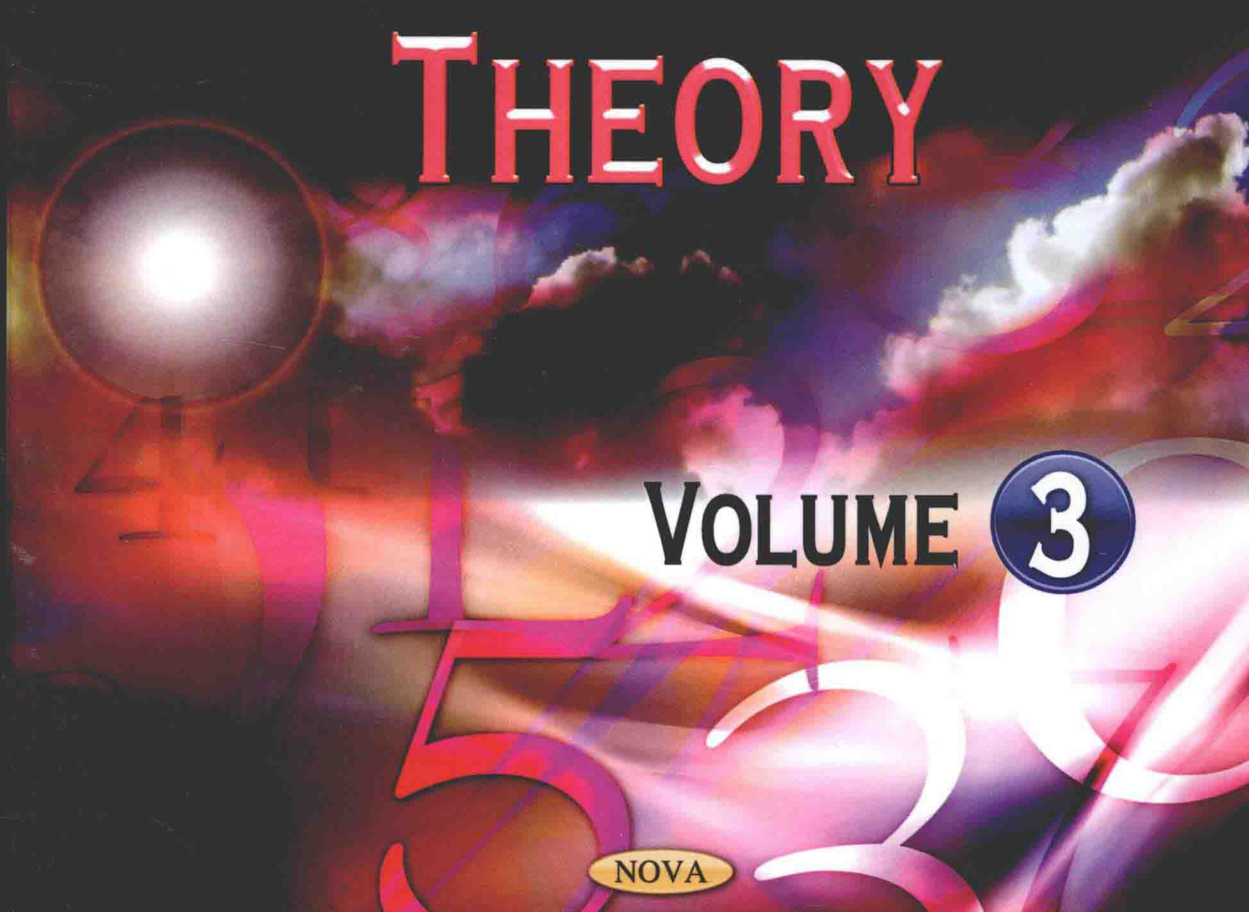
ZHI-WEI SUN

EDITOR

FRONTIERS OF COMBINATORICS AND NUMBER THEORY



VOLUME 3

NOVA

FRONTIERS OF COMBINATORICS AND NUMBER THEORY

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FRONTIERS OF COMBINATORICS AND NUMBER THEORY

VOLUME 3

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PREFACE

This book contains papers on topics in combinatorics (including graph theory) or number theory. The subject areas within correspond to the MSC (Mathematics Subject Classification) codes 05, 11, 20D60, and 52. Some topics discussed in this compilation include restricted Eisenstein series and certain convolution sums; zeroes of the Hurwitz zeta function in the interval (O,1); prime factorization conditions providing multiplicities in coset partitions of groups; mean value formulas for twisted Edwards curves; binary matrices as morphisms of a triangular category; some diophantine triples and quadruples for quadratic polynomials; codes associated with orthogonal groups; combinatorial sums and series involving inverses of the Gaussian binomial coeffecients; full friendly index sets and full product-cordial index sets of twisted cylinders; and properly charged coloring of two-dimensional arrays. (Imprint: Nova)

In Chapter 1 we parameterize the restricted Eisenstein series $E_{a,m}(q) = \sum_{\substack{n=1 \\ n \equiv a \ (\mathrm{mod}\ m)}}^{\infty} \sigma(n)q^n$ in terms of certain theta functions for $m = 8$ and then use this parameterization to evaluate the convolution sum $\sum_{\substack{m=1 \\ m \equiv a \ (\mathrm{mod}\ 8)}}^{n-1} \sigma(m)\sigma(n-m)$ for all $n \in \mathbb{N}$ and all $a \in \{0, 1, 2, 3, 4, 5, 6, 7\}$.

In 1947 Fine obtained an expression for the number $a_p(n)$ of binomial coefficients on row n of Pascal's triangle that are nonzero modulo p. In Chapter 2 we use Kummer's theorem to generalize Fine's theorem to prime powers, expressing the number $a_{p^\alpha}(n)$ of nonzero binomial coefficients modulo p^α as a sum over certain integer partitions. For fixed α, this expression can be rewritten to show explicit dependence on the number of occurrences of each subword in the base-p representation of n.

Let G be a commutative group, and assume that the order of the elements of G is at most r. Let $A \subseteq G$. In Chapter 3 we show that the set $Sym_\alpha(A) = \{h : |A \cap (A+h)| \geq \alpha|A|\}$ is a $c_1 \log|A|-$approximate group and $2Sym_\alpha(A)$ is a c_2-approximate group, where c_1 depends only on α, K, r, and c_2 depends only on α, K, where K is defined by $K = |A - A|/|A|$, Let $\Psi(n) := n\prod_{p|n}(1+\frac{1}{p})$ denote the Dedekind Ψ function. Define, for $n \geq 3$, the ratio $R(n) := \frac{\Psi(n)}{n\log\log n}$. In Chapter 4 we prove unconditionally that $R(n) < e^{\gamma}$ for $n \geq 31$. Let $N_n = 2\cdots p_n$ be the primorial of order n. We prove that the statement $R(N_n) > \frac{e^{\gamma}}{\zeta(2)}$ for $n \geq 3$ is equivalent to the Riemann Hypothesis.

In Chapter 5 the combinatorial properties of partitions with various restrictions on their hooksets are explored. A connection with numerical semigroups extends current results on simultaneous s/t-cores. Conditions that suffice for a partition to possess required hooks are developed.

For a forbidden graph L, let $ex(p; L)$ denote the maximal number of edges in a simple graph of order p not containing L. Let T_n denote the unique tree on n vertices with maximal degree $n-2$, and let $T_n^* = (V, E)$ be the tree on n vertices with $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E = \{v_0v_1, \ldots, v_0v_{n-3}, v_{n-3}v_{n-2}, v_{n-2}v_{n-1}\}$. In Chapter 6 we give exact values of $ex(p; T_n)$ and $ex(p; T_n^*)$.

In Chapter 7 we first prove an inequality for the Hurwitz zeta function $\zeta(\sigma, w)$ in the case $\sigma > 0$. As a corollary we derive that it has no zeros and is actually negative for $\sigma \in (0,1)$ and $1-\sigma \leq w$ and, as a particular instance, the known result that the classical zeta function has no zeros in $(0,1)$.

In Chapter 8 we show that the Herzog-Schönheim conjecture holds for finite groups whose orders admit certain prime factorization. A main tool here is the group-theoretic Chinese remainder theorem.

Let $p = 3$ or 5. In Chapter 9 we prove that any pair of additive forms of degree $k = p^{\tau}(p-1)$, with integer coefficients in $n > \frac{2p}{p-1}k^2 - 2k$ variables, has common p-adic zeros. The proof follows a combinatorial approach of looking for zero-sum subsequences of the sequence of all column-vectors of the 2×2 coefficient matrix.

R. Feng and H. Wu recently established a certain mean-value formula for the coordinates of the n-division points on an elliptic curve given in Weierstrass form (*A mean value formula for elliptic curves*, 2010, available at http://eprint.iacr.org/2009/586.pdf). In Chapter 10 we prove a similar result for the x and y-coordinates on a twisted Edwards elliptic curve.

A composition of binary matrices leads us to a triangular category of binomial type with corresponding triangular family of numbers $\{2^{k(n-k)}\}_{k\leq n}$ and with parameters $B(n) = 2^{\frac{n(n-1)}{2}}$, $n = 0, 1, 2, \cdots$. The standard reduced incidence algebra of this triangular category is the algebra of arithmetical functions with the convolution: $(f * g)(n) = \sum_{k=0}^{n} 2^{k(n-k)} f(k)g(n-k)$. This algebra is isomorphic to the algebra of formal power series $C[[X]]$. In Chapter 11 a distributive type characterization of binomial-multiplicative arithmetical functions is established.

In Chapter 12, we give some new examples of polynomial $D(n)$-triples and quadruples, i.e. sets of polynomials with integer coefficients, such that the product of any two of them plus a polynomial $n \in \mathbb{Z}[X]$ is a square of a polynomial with integer coefficients. The examples illustrate various theoretical properties and constructions for a quadratic polynomial n which appeared in recent papers. One of the examples gives a partial answer to the question about number of distinct $D(n)$-quadruples if n is an integer that is the product of twin primes.

Let $\mathcal{G}\ (n; W_k)$ denote the class of non bipartite graphs on n vertices containing no wheel

W_k, and $f(n; W_k) = \max\{\mathcal{E}(G) : G \in \mathcal{G}\,(n; W_k)\}$. In Chapter 13 we determine $f(n; W_5)$ and $f(n; W_6)$ by proving that (1) $f(n; W_5) = \lfloor \frac{n^2}{4} \rfloor + \lfloor \frac{s}{2} \rfloor$ for $n \geq 3$ where $s = n$ if $n \neq 4k + 2$ and $s = n - 1$ if $n = 4k + 2$ and (2) $f(n; W_6) = \lfloor \frac{n^2}{3} \rfloor$ for $n \geq 6$.

In Chapter 14 we show that the Diophantine pair $\{1, 3\}$ can not be extended to a Diophantine quintuple in the ring $\mathbb{Z}[\sqrt{-2}]$. This result completes the work of the first author and establishes nonextensibility of the Diophantine pair $\{1, 3\}$ in $\mathbb{Z}[\sqrt{-d}]$ for all $d \in \mathbb{N}$.

In Chapter 15 the number of representations of a positive integer by each of the five forms

$$x_1^2 + x_2^2 + \cdots + x_{2r}^2 + 3x_{2r+1}^2 + 3x_{2r+2}^2 + \cdots + 3x_{12}^2,\ r = 1, 2, 3, 4, 5$$

is determined.

In Chapter 16, we construct three binary linear codes $C(SO^-(2, q))$, $C(O^-(2, q))$, $C(SO^-(4, q))$, respectively associated with the orthogonal groups $SO^-(2, q)$, $O^-(2, q)$, $SO^-(4, q)$, with q powers of two. Then we obtain recursive formulas for the power moments of Kloosterman and 2-dimensional Kloosterman sums in terms of the frequencies of weights in the codes. This is done via Pless power moment identity and by utilizing the explicit expressions of Gauss sums for the orthogonal groups.

In Chapter 17 several combinatorial sums and series involving inverses of the binomial coefficients that can be evaluated in closed form are extended to the framework of q-calculus. The main tool in the proofs is a representation of the reciprocal of a Gaussian binomial coefficient as a q-Beta integral.

Let $G = (V, E)$ be a connected simple graph. A labeling $f : V \to \mathbb{Z}_2$ induces two edge labelings $f^+, f^* : E \to \mathbb{Z}_2$ defined by $f^+(xy) = f(x) + f(y)$ and $f^*(xy) = f(x)f(y)$ for each $xy \in E$. For $i \in \mathbb{Z}_2$, let $v_f(i) = |f^{-1}(i)|$, $e_{f^+}(i) = |(f^+)^{-1}(i)|$ and $e_{f^*}(i) = |(f^*)^{-1}(i)|$. A labeling f is called friendly if $|v_f(1) - v_f(0)| \leq 1$. For a friendly labeling f of a graph G, the friendly index of G under f is defined by $i_f^+(G) = e_{f^+}(1) - e_{f^+}(0)$. The set $\{i_f^+(G) \mid f \text{ is a friendly labeling of } G\}$ is called the full friendly index set of G. Also, the product-cordial index of G under f is defined by $i_f^*(G) = e_{f^*}(1) - e_{f^*}(0)$. The set $\{i_f^*(G) \mid f \text{ is a friendly labeling of } G\}$ is called the full product-cordial index set of G. In Chapter 18, we will determine full friendly index sets and full product-cordial index sets of twisted cylinders.

Let m, n, and c be fixed positive integers. A *properly charged c-coloring* of an $m \times n$ array is a coloring which uses all c colors and satisfies two additional conditions, one of which is called the *Bulge Rule*. The genesis of these conditions is related to connectivity constraints inherent in the definition of *proper array* [JQ3]. If the c colors appear in a particular order, the properly charged c-coloring is said to be *canonical*. Chapter 19 discusses two ways of enumerating the set of all properly charged canonical c-colorings of an $m \times n$ array. The first method is a generating function constructed via the transition matrix [Z] while the second method uses properties of Stirling numbers of the second kind to explicit construct each properly charged canonical c-coloring as an array of two arrays.

CONTENTS

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Chapter 1

RESTRICTED EISENSTEIN SERIES AND CERTAIN CONVOLUTION SUMS

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Abstract

We parameterize the restricted Eisenstein series $E_{a,m}(q) = \sum_{\substack{n=1 \\ n \equiv a \pmod{m}}}^{\infty} \sigma(n) q^n$ in terms of certain theta functions for $m = 8$ and then use this parameterization to evaluate the convolution sum $\sum_{\substack{m=1 \\ m \equiv a \pmod{8}}}^{n-1} \sigma(m)\sigma(n-m)$ for all $n \in \mathbb{N}$ and all $a \in \{0,1,2,3,4,5,6,7\}$.

Keywords: Convolution sums, sum of divisors function, theta functions, Eisenstein series

2000 Mathematics Subject Classification: 11A25, 11F27

1. Introduction

Throughout this paper q denotes a complex variable satisfying $|q| < 1$. Let $a \in \mathbb{Z}$ and $m \in \mathbb{N}$. We define the restricted Eisenstein series $E_{a,m}(q)$ by

$$E_{a,m}(q) := \sum_{\substack{n=1 \\ n \equiv a \pmod{m}}}^{\infty} \sigma(n) q^n,$$

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where $\sigma(n)$ denotes the sum of the positive divisors of n. An easy calculation shows that for all $a,b \in \mathbb{Z}$ and $m \in \mathbb{N}$ we have

$$E_{a,m}(q)E_{b-a,m}(q) = \sum_{\substack{n=1 \\ n \equiv b \pmod{m}}}^{\infty} S_{a,m}(n)q^n, \tag{1.1}$$

where the convolution sum $S_{a,m}(n)$ is given by

$$S_{a,m}(n) := \sum_{\substack{k=1 \\ k \equiv a \pmod{m}}}^{n-1} \sigma(k)\sigma(n-k), \quad n \in \mathbb{N}. \tag{1.2}$$

Thus, we can evaluate $S_{a,m}(n)$ for all $n \in \mathbb{N}$ with $n \equiv b \pmod{m}$ if we can determine the power series expansion of $E_{a,m}(q)E_{b-a,m}(q)$ in powers of q in a different way. We show that this can be done in the case $m = 8$ by parameterizing $E_{a,8}(q)$ in terms of the theta functions. We remark that the sums $S_{a,m}(n)$ have been evaluated for $m = 1$ in [6, eq. (3.10)], for $m = 2$ in [6, eqs. (5.3), (5.4)], for $m = 3$ in [7, Theorem 1.2] and for $m = 4$ in [4, Theorem 1.1]. We prove the following theorem in Section 3. For brevity we abbreviate $S_{a,8}(n)$ to $S_a(n)$.

Theorem. *Let $n \in \mathbb{N}$. For $k \in \mathbb{N}$ define*

$$\sigma_k(n) := \sum_{\substack{d \in \mathbb{N} \\ d \mid n}} d^k, \quad \sigma_1(n) = \sigma(n), \quad \sigma_k(t) = 0, \quad \text{if } t \notin \mathbb{N},$$

and

$$E_k = E_k(q) := \prod_{n=1}^{\infty}(1-q^{kn}).$$

Define integers $c(n), d(n), e(n)$ $(n \in \mathbb{N})$ by

$$\sum_{n=1}^{\infty} c(n)q^n := qE_2^4E_4^4, \quad \sum_{n=1}^{\infty} d(n)q^n := q^2E_8^{10}E_{16}^{-2},$$
$$\sum_{n=1}^{\infty} e(n)q^n := q^3E_2^{-4}E_4^{12}E_8^{-4}E_{16}^4.$$

(i) *If $n \equiv 0 \pmod{8}$, then*

$$S_0(n) = \frac{85}{1536}\sigma_3(n) + \frac{185}{512}\sigma_3(n/2) + \left(\frac{1}{12} - \frac{1}{2}n\right)\sigma(n), \tag{1.3}$$

$$S_1(n) = S_7(n) = \frac{1}{32}\sigma_3(n) - \frac{1}{32}\sigma_3(n/2) - 8c(n/8), \tag{1.4}$$

$$S_2(n) = S_6(n) = \frac{9}{128}\sigma_3(n) - \frac{9}{128}\sigma_3(n/2), \tag{1.5}$$

$$S_3(n) = S_5(n) = \frac{1}{32}\sigma_3(n) - \frac{1}{32}\sigma_3(n/2) + 8c(n/8), \tag{1.6}$$

$$S_4(n) = \frac{49}{512}\sigma_3(n) - \frac{49}{512}\sigma_3(n/2). \tag{1.7}$$

(ii) *If* $n \equiv 1 \pmod 8$, *then*

$$S_0(n) = S_1(n) = \frac{23}{384}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n) + \frac{19}{128}c(n), \tag{1.8}$$

$$S_2(n) = S_7(n) = \frac{3}{64}\sigma_3(n) - \frac{3}{64}c(n) + \frac{3}{2}e(n), \tag{1.9}$$

$$S_3(n) = S_6(n) = \frac{3}{64}\sigma_3(n) - \frac{3}{64}c(n) - \frac{3}{2}e(n), \tag{1.10}$$

$$S_4(n) = S_5(n) = \frac{7}{128}\sigma_3(n) - \frac{7}{128}c(n). \tag{1.11}$$

(iii) *If* $n \equiv 2 \pmod 8$, *then*

$$S_0(n) = S_2(n) = \frac{23}{32}\sigma_3(n/2) + \left(\frac{1}{8} - \frac{3}{4}n\right)\sigma(n/2) + \frac{21}{32}c(n/2), \tag{1.12}$$

$$S_1(n) = \frac{1}{4}\sigma_3(n/2) + \frac{1}{4}c(n/2) + \frac{1}{4}d(n), \tag{1.13}$$

$$S_3(n) = S_7(n) = \frac{1}{4}\sigma_3(n/2) - \frac{1}{4}c(n/2), \tag{1.14}$$

$$S_4(n) = S_6(n) = \frac{21}{32}\sigma_3(n/2) - \frac{21}{32}c(n/2), \tag{1.15}$$

$$S_5(n) = \frac{1}{4}\sigma_3(n/2) + \frac{1}{4}c(n/2) - \frac{1}{4}d(n). \tag{1.16}$$

(iv) *If* $n \equiv 3 \pmod 8$, *then*

$$S_0(n) = S_3(n) = \frac{23}{384}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n) - \frac{37}{128}c(n), \tag{1.17}$$

$$S_1(n) = S_2(n) = \frac{3}{64}\sigma_3(n) - \frac{3}{64}c(n) + \frac{3}{2}e(n), \tag{1.18}$$

$$S_4(n) = S_7(n) = \frac{7}{128}\sigma_3(n) + \frac{49}{128}c(n), \tag{1.19}$$

$$S_5(n) = S_6(n) = \frac{3}{64}\sigma_3(n) - \frac{3}{64}c(n) - \frac{3}{2}e(n). \tag{1.20}$$

(v) *If* $n \equiv 4 \pmod 8$, *then*

$$S_0(n) = S_4(n) = \frac{161}{24}\sigma_3(n/4) + \left(\frac{7}{24} - \frac{7}{4}n\right)\sigma(n/4), \tag{1.21}$$

$$S_1(n) = S_3(n) = 2\sigma_3(n/4) + 2(-1)^{(n-4)/8}c(n/4), \tag{1.22}$$

$$S_2(n) = \frac{9}{2}\sigma_3(n/4) + \frac{9}{2}c(n/4), \tag{1.23}$$

$$S_5(n) = S_7(n) = 2\sigma_3(n/4) - 2(-1)^{(n-4)/8}c(n/4), \tag{1.24}$$

$$S_6(n) = \frac{9}{2}\sigma_3(n/4) - \frac{9}{2}c(n/4). \tag{1.25}$$

(vi) *If* $n \equiv 5 \pmod 8$, *then*

$$S_0(n) = S_5(n) = \frac{23}{384}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n) + \frac{19}{128}c(n), \tag{1.26}$$

$$S_1(n) = S_4(n) = \frac{7}{128}\sigma_3(n) - \frac{7}{128}c(n), \tag{1.27}$$

$$S_2(n) = S_3(n) = \frac{3}{64}\sigma_3(n) - \frac{3}{64}c(n) + \frac{3}{2}e(n), \tag{1.28}$$

$$S_6(n) = S_7(n) = \frac{3}{64}\sigma_3(n) - \frac{3}{64}c(n) - \frac{3}{2}e(n). \tag{1.29}$$

(vii) *If* $n \equiv 6 \pmod 8$, *then*

$$S_0(n) = S_6(n) = \frac{23}{32}\sigma_3(n/2) + \left(\frac{1}{8} - \frac{3}{4}n\right)\sigma(n/2) + \frac{21}{32}c(n/2), \tag{1.30}$$

$$S_1(n) = S_5(n) = \frac{1}{4}\sigma_3(n/2) + \frac{1}{4}c(n/2), \tag{1.31}$$

$$S_2(n) = S_4(n) = \frac{21}{32}\sigma_3(n/2) - \frac{21}{32}c(n/2), \tag{1.32}$$

$$S_3(n) = \frac{1}{4}\sigma_3(n/2) - \frac{1}{4}c(n/2) + 8e(n/2), \tag{1.33}$$

$$S_7(n) = \frac{1}{4}\sigma_3(n/2) - \frac{1}{4}c(n/2) - 8e(n/2). \tag{1.34}$$

(viii) *If* $n \equiv 7 \pmod 8$, *then*

$$S_0(n) = S_7(n) = \frac{23}{384}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma(n) - \frac{37}{128}c(n), \tag{1.35}$$

$$S_1(n) = S_6(n) = \frac{3}{64}\sigma_3(n) - \frac{3}{64}c(n) - \frac{3}{2}e(n), \tag{1.36}$$

$$S_2(n) = S_5(n) = \frac{3}{64}\sigma_3(n) - \frac{3}{64}c(n) + \frac{3}{2}e(n), \tag{1.37}$$

$$S_3(n) = S_4(n) = \frac{7}{128}\sigma_3(n) + \frac{49}{128}c(n). \tag{1.38}$$

We remark that

$$c(n) = e(n) = 0, \text{ if } n \equiv 0 \pmod 2, \quad d(n) = 0, \text{ if } n \not\equiv 2 \pmod 8.$$

There are 8 sums $S_a(n)$ $(a = 0,1,2,\ldots,7)$ to be evaluated in 8 cases depending upon $n \pmod 8$. Thus there are $8 \times 8 = 64$ formulae in total. It is interesting to note that of these 64 formulae just 6 require only the divisor functions σ_3 and σ, while of the remaining 58 formulae, in addition to the divisor functions σ_3 and σ, 38 require c, 18 require c and e, and 2 require c and d. It is simple to check that no linear relation of the type

$$d(n) = \big(A + B(-1)^{(n-2)/8}\big)\sigma_3(n/2) + \big(C + D(-1)^{(n-2)/8}\big)c(n/2) \\ + \big(E + F(-1)^{(n-2)/8}\big)e(n/2), \; n \equiv 2 \pmod 8, \tag{1.39}$$

exists so that d is really required.

In proving our theorem we make use of the following result of Williams [8, Theorem 1]

$$\sum_{\substack{m \in \mathbb{N} \\ m < n/8}} \sigma(m)\sigma(n-8m) = \frac{1}{192}\sigma_3(n) + \frac{1}{64}\sigma_3(n/2) + \frac{1}{16}\sigma_3(n/4) + \frac{1}{3}\sigma_3(n/8) \\ + \Big(\frac{1}{24} - \frac{1}{32}n\Big)\sigma(n) + \Big(\frac{1}{24} - \frac{1}{4}n\Big)\sigma(n/8) - \frac{1}{64}c(n).$$

2. Parameterization of Theta Functions and Restricted Eisenstein Series

The theta functions $\varphi(q)$ and $\psi(q)$ are defined by

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \quad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}, \quad q \in \mathbb{C}, \; |q| < 1. \tag{2.1}$$

The basic properties of these functions are

$$\begin{cases} \varphi(q) + \varphi(-q) = 2\varphi(q^4), & [5, \text{eq. } (3.6.1)], \\ \varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2), & [5, \text{eq. } (3.6.7)], \\ \varphi(q)\varphi(-q) = \varphi^2(-q^2), & [5, \text{eq. } (1.3.32)], \\ \varphi(q) - \varphi(-q) = 4q\psi(q^8), & [5, \text{eq. } (3.6.2)]. \end{cases} \tag{2.2}$$

Using the first three of these relations we can parameterize $\varphi(\pm q), \varphi(\pm q^2), \varphi(\pm q^4)$, $\varphi(\pm q^8)$ and $\varphi(\pm q^{16})$ in terms of the parameters A, B and X defined by

$$A = A(q) := \varphi(q), \; B = B(q) := \varphi(-q), \; X = X(q) := \frac{1}{2}AB(A^2 + B^2), \tag{2.3}$$

namely

$$\begin{cases} \varphi(q) = A, \ \varphi(-q) = B, \\ \varphi(q^2) = \left(\dfrac{A^2+B^2}{2}\right)^{\frac{1}{2}}, \ \varphi(-q^2) = (AB)^{\frac{1}{2}}, \\ \varphi(q^4) = \dfrac{1}{2}(A+B), \\ \varphi(-q^4) = X^{\frac{1}{4}}, \\ \varphi(q^8) = \dfrac{1}{2}\left(\left(\dfrac{A^2+B^2}{2}\right)^{\frac{1}{2}} + (AB)^{\frac{1}{2}}\right), \\ \varphi(-q^8) = \left(\dfrac{A+B}{2}\right)^{\frac{1}{2}} X^{\frac{1}{8}}, \\ \varphi(q^{16}) = \dfrac{1}{2}\left(\dfrac{A+B}{2} + X^{\frac{1}{4}}\right). \end{cases} \tag{2.4}$$

For future use, we note that under the transformation $q \mapsto -q$, we have

$$A(-q) = B, \ B(-q) = A, \ X(-q) = X, \tag{2.5}$$

and under the transformation $q \mapsto q^2$, we have

$$\begin{cases} A(q^2) = \left(\dfrac{A^2+B^2}{2}\right)^{\frac{1}{2}}, \ B(q^2) = (AB)^{\frac{1}{2}}, \\ A(q^2)B(q^2) = X^{\frac{1}{2}}, \\ A^2(q^2) + B^2(q^2) = \dfrac{(A+B)^2}{2}, \\ A^2(q^2) - B^2(q^2) = \dfrac{(A-B)^2}{2}, \\ (A(q^2)+B(q^2))^2 = \dfrac{(A+B)^2}{2} + 2X^{\frac{1}{2}}, \\ (A(q^2)-B(q^2))^2 = \dfrac{(A+B)^2}{2} - 2X^{\frac{1}{2}}, \\ X(q^2) = \dfrac{(A+B)^2}{4} X^{\frac{1}{2}}. \end{cases} \tag{2.6}$$

We recall next how $E_{1,2}(q) = \sum_{\substack{n=1 \\ n \equiv 1 \pmod 2}}^{\infty} \sigma(n) q^n$ can be parameterized in terms of A and B. Jacobi proved that the number $r_4(n)$ of representations of an odd positive integer n as the sum of four integral squares is given by $r_4(n) = 8\sigma(n)$. Hence

$$\begin{aligned} A^4 - B^4 &= \varphi^4(q) - \varphi^4(-q) = \sum_{n=0}^{\infty} r_4(n) q^n - \sum_{n=0}^{\infty} r_4(n)(-q)^n \\ &= 2 \sum_{\substack{n=1 \\ n \equiv 1 \pmod 2}}^{\infty} r_4(n) q^n = 16 \sum_{\substack{n=1 \\ n \equiv 1 \pmod 2}}^{\infty} \sigma(n) q^n \end{aligned}$$