# 国外数学名著系列（续一）

（影印版）66

Yu. V. Egorov　M. A. Shubin (Eds.)

# Partial Differential Equations IV

## Microlocal Analysis and Hyperbolic Equations

# 偏微分方程 IV

## 微局部分析和双曲型方程

科学出版社
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北 京

图字：01-2008-5381

Yu. V. Egorov, M. A. Shubin(Eds.): Partial Differential Equations IV: Microlocal Analysis and Hyperbolic Equations


**图书在版编目(CIP)数据**

偏微分方程 IV：微局部分析和双曲型方程=Partial Differential Equations IV: Microlocal Analysis and Hyperbolic Equations/(俄罗斯)叶戈罗夫(Egorov, Y. V.)等编著. —影印版. —北京：科学出版社，2009

(国外数学名著系列；66)

ISBN 978-7-03-023509-1

I. 偏… II. 叶… III. ①偏微分方程-英文 ②双曲型方程-英文 IV. O175.2

中国版本图书馆 CIP 数据核字(2008) 第 186169 号

责任编辑：范庆奎/责任印刷：钱玉芬/封面设计：黄华斌

科学出版社 出版
北京东黄城根北街 16 号
邮政编码：100717
http://www.sciencep.com

双青印刷厂 印刷

科学出版社发行 各地新华书店经销

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2009 年 1 月第 一 版 开本：B5(720×1000)
2009 年 1 月第一次印刷 印张：16
印数：1—2 500 字数：304 000

**定价：58.00 元**

(如有印装质量问题，我社负责调换〈双青〉)

# 《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

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# I. Microlocal Analysis

## Yu.V. Egorov

Translated from the Russian
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## Contents

# Preface

The microlocal analysis is the local analysis in cotangent bundle space. The remarkable progress made in the theory of linear partial differential equations over the past two decades is essentially due to the extensive applicaton of the microlocalization idea. The Hamiltonian systems, canonical transformations, Lagrange manifolds and other concepts, used in theoretical mechanics for examining processes in the phase space, have in recent years become the central objects of the theory of differential equations. For example, the evolution of singularities of solutions of a differential equation is described most naturally in terms of Lagrange manifolds and Hamiltonian systems, the solvability conditions are formulated in terms of the behaviour of integral curves of the Hamiltonian system whose Hamiltonian function serves as the characteristic form, the class of pseudodifferential equations arises in a natural way from that of differential equations under the action of canonical transformations, the class of subelliptic operators is defined by means of the Poisson brackets, etc. The difficulty faced in the microlocal analysis is connected with the principle of uncertainty which does not permit us to localize a function in any neighbourhood of a point of the cotangent space.

This paper presents a survey of the most interesting results, from our point of view, of the microlocal analysis achieved over the recent years. Unfortunately, due to lack of space many significant results could not be included. Also incomplete is the list of the literature cited; more complete lists can be found in Egorov [1984], Shubin [1978], Hörmander [1963, 1983, 1985], Taylor [1981] and Trèves [1982].

The author expresses his thanks to V.Ya. Ivrii for his useful critical comments.

# Chapter 1
# Microlocal Properties of Distributions

## § 1. Microlocalization

The study of the singularities of solutions constitutes one of the most important problems in the theory of differential equations. In this theory, just as in other mathematical disciplines, one often examines functions modulo smooth ones, so that the points where a given function is infinitely differentiable may be neglected. This approach reflects physical realities: singular points correspond to those phenomena which are most interesting from the point of view of each physical theory.

In investigating physical processes that take place in a bounded space an extensive use is made of the principle of *locality*. Its essence lies in that by knowing the state of the process at a given moment of time in a fixed region $\Omega$ of the physical space one may determine, by means of physical laws, the course of the process in a region $\Omega'$, lying strictly inside $\Omega$, for a future time interval. During this time interval the effect of processes taking place outside $\Omega$ will have no influence on phenomena in $\Omega'$ because the effect is propagated with a fintie velocity.

We can introduce a more general principle, the principle of *microlocality*, by examining the phenomenon in a bounded region of the phase space. If we know the state of the process in this region at a certain moment of time, we can describe this process for future close points lying strictly inside the region. In physical terms, this means that the change in the impulse too takes place with a finite velocity because the acting forces are finite.

The above-mentioned principles are reflected in mathematical physics in the investigation of singularities of solutions of differential equations. Namely, local properties of such solutions are those properties which remain unaltered when the solutions are multiplied by smooth functions with a small support. Microlocal properties of a solution refer naturally to those properties which do not change on "multiplication" of the solution by a smooth function having support in a small neighbourhood of the given point in the phase space. However, this operation is much more complicated. In fact, it consists in multiplying by an ordinary smooth cutoff function with a small support, in applying Fourier transformation, in multiplying successively by a smooth cutoff function of dual coordinates, and in applying inverse Fourier transformation. Instead of Fourier transformation we can also use some other decomposition in plane waves; for example, the Radon transformation. In fact, the microlocal analysis is the local analysis on the cotangent bundle space.

A special feature of the microlocal analysis is the fact that localization in the phase space is possible only to a certain extent: the localization of spatial coordinates obstructs that of impulses. In quantum mechanics, this fact is referred to as the Heisenberg uncertainty principle.

The last two decades have seen immensely fruitful applications of the microlocality principle to the theory of partial differential equations. Every function (ordinary or generalized) can be regarded as an aggregate of linear differential equations which this functon satisfies. The microlocality principle extends in a natural manner this aggregate to the system of pseudodifferential equations derived from differential equations by transforming the phase space without altering its structure. By applying the microlocality principle we not only obtain a more precise description of singular points of a distribution but we also have a simpler description of the propagation process of these singularities. By this principle we can also extend to distributions the operations defined initially for smooth functions only; for example, the operation of taking trace or the operation of multiplication, etc.

Let us explain the idea of microlocalization with the following simple example. Let $n \geqslant 2$ be a natural number and let $f$ be a function in $\mathbb{R}^n$ of the form $f(x) = g(\alpha \cdot x)$, where $\alpha \in \mathbb{R}^n \backslash 0$, $\alpha \cdot x = \sum_{j=1}^n \alpha_j \cdot x_j$ and $g$ is a function of a single variable. If $g(t)$ has a singularity, for example, if $g(t)$ is not differentiable at $t = t_0$, then all the points $x$ lying on the plane $\alpha \cdot x = t_0$ are singular points of $f$. However, $f$ is a smooth function in each direction lying on this plane so that for it singular will be only the direction of the vector $\alpha$. Radon's theorem enables us to represent each distribution $f$ in $\mathscr{D}'(\mathbb{R}^n)$ as an integral of plane waves:

$$f(x) = \int_{|\alpha|=1} g_\alpha(\alpha \cdot x)\, d\alpha.$$

Therefore at each point $x$ those directions $\alpha$ will be singular for which the distribution $g_\alpha(t)$ has a singularity at the point $t = x \cdot \alpha$. If, instead of Radon's theorem, we apply Fourier transformation, then $f$ can be represented as an integral of plane waves:

$$f(x) = \int g(\alpha) e^{i\alpha \cdot x}\, d\alpha,$$

where the integration is performed over the whole $\mathbb{R}^n$. Now those directions of $\alpha$ become singular for $f$ in which $g(t\alpha)$ does not decrease, as $t \to \infty$, rapidly enough.

As mentioned earlier, in the modern theory of differential equations the microlocality principle is extensively applied to investigate the singularities of the solution. Many important results achieved in recent years by means of this principle in the theory of boundary-value problems, in the spectral theory, in the theory of functions of several complex variables, and in other branches of mathematics point towards great potentialities of the microlocal analysis.

## § 2. Wave Front of Distribution. Its Functorial Properties

**2.1. Definition of the Wave Front.** The notion of a singular point of a distribution does not have only one meaning. Depending on the problem under discussion, a singular point may signify a point of discontinuity, or a point where the function becomes infinite, or a point where the function has an essential singularity in the sense of the theory of complex variables, etc. For the general theory of distributions, the most natural is the following

**Definition 2.1.** A point $x_0$ is a non-singular point for a distribution $u$ if there exists a function $\varphi$ in $C_0^\infty(\mathbb{R}^n)$ such that $\varphi(x_0) \neq 0$ and $\varphi u \in C^\infty(\mathbb{R}^n)$.

It follows from the definition that the singular points of a distribution constitute a closed set. This set is known as the *singular support* of the distribution $u$ and is denoted by sing supp $u$. We can easily see that this set is invariant under diffeomorphisms of the space; thus the definition of a singular support can be readily extended to distributions on a smooth manifold.

For a smooth manifold $X$, we denote by $T^*(X)$ the cotangent bundle space and by $T^*(X)\backslash 0$ the same space with the zero section removed (see, for example, Arnol'd [1974], Egorov [1984]). The following definition and examples are due to Hörmander. It should be remarked that in the general theory there are many close concepts that are extensively used, namely, the analytic wave front, Gevrey wave front, oscillation front, etc (see Hörmander [1983, 1985], Trèves [1982], and § 4.2 of Chapter 6).

**Definition 2.2.** A point $(x_0, \xi_0) \in T^*(\mathbb{R}^n)\backslash 0$ does not belong to the *wave front* of a distribution $u$ in $\mathscr{D}'(\mathbb{R}^n)$ if there are a function $\varphi$ in $C_0^\infty(\mathbb{R}^n)$, with $\varphi(x_0) \neq 0$, and a cone $\Gamma$ in $\mathbb{R}^n$, with vertex at the origin of coordinates, which contains in its interior the ray $\{\xi; \xi = t\xi_0, t > 0\}$ such that the relations

$$|\tilde{\varphi u}(\xi)| = 0((1 + |\xi|)^{-N})$$

holds for all $\xi \in \Gamma$ and all integers $N$.

The wave front of $u$ will be denoted by WF($u$).

**Example 2.1.** If the distribution $u$ is a plane wave, that is, if $u(x) = g(\alpha \cdot x)$, where $\alpha \in \mathbb{R}^n\backslash 0$ and $g \in \mathscr{D}'(\mathbb{R})$, then every direction $\xi_0$, non-collinear with the vector $\alpha$, is non-singular for $u$. That is, WF($u$) may only contain points $(x_0, \xi_0)$ for which $\xi_0 = t\alpha$ with $t \in \mathbb{R}\backslash 0$ and $\alpha \cdot x_0 \in$ sing supp $g$.

**Example 2.2.** Suppose that $\eta \in \mathbb{R}^n\backslash 0$, $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^n)$, $\tilde{\varphi}(0) = 1$ and $\tilde{\varphi}(0) > 0$. Then for the function

$$u(x) = \sum_{k=1}^{\infty} \frac{\varphi(kx)}{k^2} e^{ik^2(x,\eta)}$$

which is continuous in $\mathbb{R}^n$, the wave front consists of the ray $\{(0, t\eta), t > 0\}$ (Hörmander [1979a]).

Summing over $\eta$, we obtain from this example a function whose wave front coincides with an arbitrary conical closed subset of $T^*(\mathbb{R}^n)\backslash 0$.

**2.2. Localization of Wave Front.** It is comparatively easy to establish the following properties of the wave front (see Hörmander [1971, 1983, 1985], Egorov [1984]):

1. If $u \in \mathscr{D}'(\mathbb{R}^n)$ and $\varphi \in C_0^\infty(\mathbb{R}^n)$, then

$$\mathrm{WF}(\varphi u) \subset \mathrm{WF}(u).$$

2. If $\pi\colon T^*(\mathbb{R}^n) \to \mathbb{R}^n$ is the natural projection, then $\pi\mathrm{WF}(u) = \operatorname{sing\,supp} u$ for every $u$ in $\mathscr{D}'(\mathbb{R}^n)$.

**2.3. Wave Front and Singularities of One-Dimensional Distributions.** Let $X$ and $Y$ be smooth manifolds, and let $f\colon X \to Y$ be a smooth map. $f$ is called *proper* if the set $f^{-1}(K)$ is compact for every compact set $K$ in $Y$. For a $\varphi \in C_0^\infty(Y)$ and a proper map $f$ we set

$$f^*\varphi(x) = \varphi(f(x)).$$

Then $f^*\varphi \in C_0^\infty(X)$ and $f^*$ is a continuous map from $\mathscr{D}(Y)$ into $\mathscr{D}(X)$. This enables us to define, by means of duality, the *pushforward* $f_*u$ of every *distribution* $u \in \mathscr{D}'(X)$ by the formula

$$\langle f_*u, \varphi\rangle = \langle u, f^*\varphi\rangle.$$

It is obvious that the same construction remains valid for an arbitrary map $f$ (not necessarily proper) when $u$ has a compact support, that is, when $u \in \mathscr{E}'(X)$.

The *pullback* $f^*u$ of a *distribution* is defined when $f$ is a submersion of $X$ onto $Y$. This last condition means that for every point $y \in Y$ the set $f^{-1}(y)$ is a smooth submanifold of $X$ and all these submanifolds are diffeomorphic to a fixed $k$-dimensional smooth manifold. Locally such a map is a projection, and in a suitable local coordinate system it becomes the projection $\mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^k$. If $\varphi \in C_0^\infty(X)$ and the support of $\varphi$ lies in a coordinate neighbourhood, and if $x = (x', x'')$, where $x' \in \mathbb{R}^k$, $x'' \in \mathbb{R}^l$, then $f_*\varphi(x') = \int \varphi(x', x'')\,dx''$. For any $u \in \mathscr{D}'(Y)$ we now set

$$\langle f^*u, \varphi\rangle = \langle u, f_*\varphi\rangle.$$

If $f$ is both proper and submersion (for example, if $f$ is a diffeomorphism), then $f^*u$ and $f_*v$ are simultaneously defined for every $u \in \mathscr{D}'(Y)$, $v \in \mathscr{D}'(X)$.

**Example 2.3.** Let $\pi\colon \mathbb{R}^n \to \mathbb{R}^1$ be the projection onto the $x_1$-axis. In this case both $\pi^*u$, $u \in \mathscr{D}'(\mathbb{R})$, and $\pi_*v$, $v \in \mathscr{E}'(\mathbb{R}^n)$, are defined, and

$$\pi^*v(x^1) = u(x^1) \otimes 1_{x_2,\ldots,x_n};$$

$$\pi_*v(x) = \int v(x_1, x_2, \ldots, x_n)\,dx_2 \ldots dx_n.$$

Using these concepts, we can establish the following

**Theorem 2.1.** *Let $\Omega$ be a domain in $\mathbb{R}^n$, and let $u \in \mathscr{D}'(\Omega)$. A point $(x_0, \xi_0)$ of $T^*(\Omega)\backslash 0$ does not lie in* $\mathrm{WF}(u)$ *if and only if there exist a function $\varphi$ in $C_0^\infty(\Omega)$, with $\varphi(x_0) \neq 0$, and an $\varepsilon > 0$ such that for every smooth function $f\colon \operatorname{supp}\varphi \to \mathbb{R}$, with $|\operatorname{grad} f(x_0) - \xi_0| < \varepsilon$, the function $f_*(\varphi u)(t)$ is infinitely differentiable on the real line* (Guillemin and Sternberg [1977], Egorov [1985a]).