Differential Geometry and Symmetric Spaces

Sigurđur Helgason

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts

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PREFACE

According to its original definition, a symmetric space is a Riemannian manifold whose curvature tensor is invariant under all parallel translations. The theory of symmetric spaces was initiated by É. Cartan in 1926 and was vigorously developed by him in the late 1920's. By their definition, symmetric spaces form a special topic in Riemannian geometry; their theory, however, has merged with the theory of semisimple Lie groups. This circumstance is the source of very detailed and extensive information about these spaces. They can therefore often serve as examples for the testing of general conjectures. On the other hand, symmetric spaces are numerous enough and their special nature among Riemannian manifolds so clear that a properly formulated extrapolation to general Riemannian manifolds often leads to good questions and conjectures.

The definition above does not immediately suggest the special nature of symmetric spaces (especially if one recalls that all Riemannian manifolds and all Kähler manifolds possess tensor fields invariant under the parallelism). However, the theory leads to the remarkable fact that symmetric spaces are locally just the Riemannian manifolds of the form $\boldsymbol{R}^n \times G/K$ where $\boldsymbol{R}^n$ is a Euclidean n-space, G is a semisimple Lie group which has an involutive automorphism whose fixed point set is the (essentially) compact group K, and G/K is provided with a G-invariant Riemannian structure. É. Cartan's classification of all real simple Lie algebras now led him quickly to an explicit classification of symmetric spaces in terms of the classical and exceptional simple Lie groups. On the other hand, the semisimple Lie group G (or rather the local isomorphism class of G) above is completely arbitrary; in this way valuable geometric tools become available to the theory of semisimple Lie groups. In addition, the theory of symmetric spaces helps to unify and explain in a general way various phenomena in classical geometries. Thus the isomorphisms which occur among the classical groups of low dimensions are geometrically interpreted by means of isometries; the analogy between the spherical geometries and the hyperbolic geometries is a special case of a general duality for symmetric spaces.

On a symmetric space with its well-developed geometry, global function theory becomes particularly interesting. Integration theory, Fourier analysis, and partial differential operators arise here in a canonical fashion by the requirement of geometric invariance. Although these subjects and their relationship are very well developed in Euclidean space (Lebesgue integral, Fourier integral, differential operators with constant coefficients) the extension to general symmetric spaces leads immediately to interesting unsolved problems. The two types of non-Euclidean symmetric spaces, the compact type and the noncompact type, offer different sorts of function-theoretic problems. The symmetric spaces of the noncompact type present no topological difficulties (the spaces being homeomorphic to Euclidean spaces) and their function theory ties up with the theory of infinite-dimensional representations of arbitrary semisimple Lie groups, which has made great progress in recent years. For the symmetric spaces of the compact type, on the other hand, the classical theory of finite-dimensional representations of compact Lie groups provides a natural framework, but the geometry of the spaces enters now in a less trivial fashion into their function theory.

The objective of the present book is to provide a self-contained introduction to Cartan's theory, as well as to more recent developments in the theory of functions on symmetric spaces.

Chapter I deals with the differential-geometric prerequisites, and the basic geometric properties of symmetric spaces are developed in Chapter IV. From then on the subject is primarily Lie group theory, and in Chapter IX Cartan's classification of symmetric spaces is presented. Although this classification may be considered as the culmination of Cartan's theory, we have confined Chapter IX to proofs of general theorems involved in the classification and to a description of Cartan's list. The justification of this notable omission is first that the usefulness of the classification for experimentation is based on its existence rather than on the proof that it exhausts the class of symmetric spaces; secondly this omission enabled us to include Chapter X (on functions on symmetric spaces) where it is felt that more open questions present themselves. At some places we indicate connections with topics in classical analysis, such as Fourier analysis, theory of special functions (Bessel, Legendre), and integral theorems for invariant differential equations. However, no account is given of the role of symmetric spaces in the theory of automorphic functions and analytic number theory, nor have we found it possible to include more recent topological investigations of symmetric spaces.

Each chapter begins with a short summary and ends with an identification of sources as well as some comments on the historical development.

The purpose of these historical notes is primarily to orient the reader in the vast literature and secondly they are an attempt to give credit where it is due, but here we must apologize in advance for incompleteness as well as possible inaccuracies.†

This book grew out of lectures given at the University of Chicago 1958 and at Columbia University 1959–1960. At Columbia I had the privilege of many long and informative discussions with Professor Harish-Chandra; large parts of Chapters VIII and X are devoted to results of his. I am happy to express here my deep gratitude to him. I am also indebted to Professors A. Korányi, K. deLeeuw, E. Luft and H. Mirkil who read large portions of the manuscript and suggested many improvements. Finally I want to thank my wife who patiently helped with the preparation of the manuscript and did all the typing.

Suggestions to the Reader

Since this book is intended for readers with varied backgrounds we give here some suggestions for its use.

Chapter I, Chapter IV, § 1, and Chapter VIII, § 1 – § 3 can be read independently of the rest of the book. These parts would give the reader an incomplete but short and elementary introduction to modern differenctial geometry, with only advanced calculus and some point-set topology as prerequisites.

Chapter I, § 1 – § 6, Chapter II, and Chapter III can be read independently of the rest of the book as an introduction to semisimple Lie groups. However, Chapters II and III assume some familiarity with the elements of the theory of topological groups.

Chapters I – IX require no further prerequisites. Chapter X, however, makes use of a few facts from Hilbert space theory and assumes some knowledge of measure theory.

Each chapter ends with a few exercises. With a few possible exceptions (indicated with a star) the exercises can be worked out with methods developed in the text. The starred exercises are theorems which might have been included in the text, but were not found necessary for the subsequent chapters.

S. Helgason

† "En hvatki es missagt es í fræðum þessum, þá es skylt at hafa þat heldr, es sannara reynisk"; [Ari Fróði: Íslendingabók (1124)].

CONTENTS

CHAPTER III

Structure of Semisimple Lie Algebras

CHAPTER IV

Symmetric Spaces

CHAPTER V

Decomposition of Symmetric Spaces

CHAPTER VI

Symmetric Spaces of the Noncompact Type

CHAPTER VII

Symmetric Spaces of the Compact Type

CHAPTER VIII

Hermitian Symmetric Spaces

CHAPTER IX

On the Classification of Symmetric Spaces

CHAPTER X

Functions on Symmetric Spaces

CHAPTER I

ELEMENTARY DIFFERENTIAL GEOMETRY

This introductory chapter divides in a natural way into three parts: §1-§3 which deal with tensor fields on manifolds, §4-§8 which treat general properties of affine connections, and §9-§14 which give an introduction to Riemannian geometry with some emphasis on topics needed for the later treatment of symmetric spaces.

§1-§3. When a Euclidean space is stripped of its vector space structure and only its differentiable structure retained, there are many ways of piecing together domains of it in a smooth manner, thereby obtaining a so-called differentiable manifold. Local concepts like a differentiable function and a tangent vector can still be given a meaning whereby the manifold can be viewed "tangentially," that is, through its family of tangent spaces as a curve in the plane is, roughly speaking, determined by its family of tangents. This viewpoint leads to the study of tensor fields, which are important tools in local and global differential geometry. They form an algebra $\mathfrak{D}(M)$, the mixed tensor algebra over the manifold M. The alternate covariant tensor fields (the differential forms) form a submodule $\mathfrak{A}(M)$ of $\mathfrak{D}(M)$ which inherits a multiplication from $\mathfrak{D}(M)$, the exterior multiplication. The resulting algebra is called the Grassmann algebra of M. Through the work of É. Cartan the Grassmann algebra with the exterior differentiation d has become an indispensable tool for dealing with submanifolds, these being analytically described by the zeros of differential forms. Moreover, the pair $(\mathfrak{A}(M), d)$ determines the cohomology of M via de Rham's theorem, which however will not be dealt with here.

§4-§8. The concept of an affine connection was first defined by Levi-Civita for Riemannian manifolds, generalizing significantly the notion of parallelism for Euclidean spaces. On a manifold with a countable basis an affine connection always exists (see the exercises following this chapter). Given an affine connection on a manifold M there is to each curve $\gamma(t)$ in M associated an isomorphism between any two tangent spaces $M_{\gamma(t_1)}$ and $M_{\gamma(t_2)}$. Thus, an affine connection makes it possible to relate tangent spaces at distant points of the manifold. If the tangent vectors of the curve $\gamma(t)$ all correspond under these isomorphisms we have the analog of a straight line, the so-called geodesic. The theory of affine connections mainly amounts to a study of the mappings $\text{Exp}_p : M_p \to M$ under which straight lines (or segments of them) through the origin in the tangent space M_p correspond to geodesics through p in M. Each mapping Exp_p is a diffeomorphism of a neighborhood of 0 in M_p into M, giving the so-called normal coordinates at p. Some other local properties of Exp_p are given in §6, the existence of convex neighborhoods and a formula for the differential of Exp_p.

An affine connection gives rise to two important tensor fields, the curvature tensor field and the torsion tensor field which in turn describe the affine connection through É. Cartan's structural equations [(6) and (7), §8].

§9-§14. A particularly interesting tensor field on a manifold is the so-called Riemannian structure. This gives rise to a metric on the manifold in a canonical fashion. It also determines an affine connection on the manifold, the Riemannian connection; this affine connection has the property that the geodesic forms the shortest curve between any two (not too distant) points. The relation between the metric and geodesics is further developed in §9-§10. The treatment is mainly based on the structural equations of É. Cartan and is independent of the Calculus of Variations.

The higher-dimensional analog of the Gaussian curvature of a surface was discovered by Riemann. Riemann introduced a tensor field which for any pair of tangent vectors at a point measures the corresponding sectional curvature, that is, the Gaussian curvature of the surface generated by the geodesics tangent to the plane spanned by the two vectors. Of particular interest are Riemannian manifolds for which the sectional curvature always has the same sign. The irreducible symmetric spaces are of this type. Riemannian manifolds of negative curvature are considered in §13 owing to their importance in the theory of symmetric spaces. Much progress has been made recently in the study of Riemannian manifolds whose sectional curvature is bounded from below by a constant > 0. However, no discussion of these is given since it is not needed in later chapters. The last section deals with totally geodesic submanifolds which are characterized by the condition that a geodesic tangent to the submanifold at a point lies entirely in it. In contrast to the situation for general Riemannian manifolds, totally geodesic submanifolds are a common occurrence for symmetric spaces.

§1. Manifolds

Let $\boldsymbol{R}^m$ and $\boldsymbol{R}^n$ denote two Euclidean spaces of m and n dimensions, respectively. Let O and O' be open subsets, $O \subset \boldsymbol{R}^m$, $O' \subset \boldsymbol{R}^n$ and suppose φ is a mapping of O into O'. The mapping φ is called *differentiable* if the coordinates $y_j(\varphi(p))$ of $\varphi(p)$ are differentiable (that is, indefinitely differentiable) functions of the coordinates $x_i(p)$, $p \in O$. The mapping φ is called *analytic* if for each point $p \in O$ there exists a neighborhood U of p and n power series P_j $(1 \leqslant j \leqslant n)$ in m variables such that $y_j(\varphi(q)) = P_j(x_1(q) - x_1(p), \ldots, x_m(q) - x_m(p))$ $(1 \leqslant j \leqslant n)$ for $q \in U$. A differentiable mapping $\varphi : O \rightarrow O'$ is called *a diffeomorphism* of O onto O' if $\varphi(O) = O'$, φ is one-to-one, and the inverse mapping φ^{-1} is differentiable. In the case when $n = 1$ it is customary to replace the term "mapping" by the term "function."

An analytic function on $\boldsymbol{R}^m$ which vanishes on an open set is identically 0. For differentiable functions the situation is completely different. In fact, if A and B are disjoint subsets of $\boldsymbol{R}^m$, A compact and B closed, then there exists a differentiable function φ which is identically 1 on A and identically 0 on B. The standard procedure for constructing such a function φ is as follows:

Let $0 < a < b$ and consider the function f on $\boldsymbol{R}$ defined by

$$f(x) = \begin{cases} \exp\left(\dfrac{1}{x-b} - \dfrac{1}{x-a}\right) & \text{if } a < x < b, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is differentiable and the same holds for the function

$$F(x) = \int_x^b f(t)\,dt \Big/ \int_a^b f(t)\,dt,$$

which has value 1 for $x \leqslant a$ and 0 for $x \geqslant b$. The function ψ on $\boldsymbol{R}^m$ given by

$$\psi(x_1, \ldots, x_m) = F(x_1^2 + \ldots + x_m^2)$$

is differentiable and has values 1 for $x_1^2 + \ldots + x_m^2 \leqslant a$ and 0 for $x_1^2 + \ldots + x_m^2 \geqslant b$. Let S and S' be two concentric spheres in $\boldsymbol{R}^m$, S' lying inside S. Starting from ψ we can by means of a linear transformation of $\boldsymbol{R}^m$ construct a differentiable function on $\boldsymbol{R}^m$ with value 1 in the interior of S' and value 0 outside S. Turning now to the sets A and B we can, owing to the compactness of A, find finitely many spheres S_i $(1 \leqslant i \leqslant n)$, such that the corresponding open balls B_i $(1 \leqslant i \leqslant n)$, form a covering of A (that is, $A \subset \bigcup_{i=1}^n B_i$) and such that the closed balls $\bar{B}_i$ $(1 \leqslant i \leqslant n)$ do not intersect B. Each sphere S_i can be shrunk to a concentric sphere S_i' such that the corresponding open balls B_i' still form a covering of A. Let ψ_i be a differentiable function on $\boldsymbol{R}^m$ which is identically 1 on B_i' and identically 0 in the complement of B_i. Then the function

$$\varphi = 1 - (1 - \psi_1)(1 - \psi_2) \ldots (1 - \psi_n)$$

is a differentiable function on $\boldsymbol{R}^m$ which is identically 1 on A and identically 0 on B.

Let M be a topological space. We assume that M satisfies the Hausdorf separation axiom which states that any two different points in M can be separated by disjoint open sets. An *open chart* on M is a pair (U, φ) where U is an open subset of M and φ is a homeomorphism of U onto an open subset of $\boldsymbol{R}^m$.

Definition. Let M be a Hausdorf space. A *differentiable structure* on M of dimension m is a collection of open charts $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ on M where $\varphi_\alpha(U_\alpha)$ is an open subset of $\boldsymbol{R}^m$ such that the following conditions are satisfied:

(M_1) $M = \bigcup_{\alpha \in A} U_\alpha$.

(M_2) For each pair $\alpha, \beta \in A$ the mapping $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a differentiable mapping of $\varphi_\alpha(U_\alpha \cap U_\beta)$ onto $\varphi_\beta(U_\alpha \cap U_\beta)$.

(M_3) The collection $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ is a maximal family of open charts for which (M_1) and (M_2) hold.

A *differentiable manifold* (or C^∞ *manifold* or simply *manifold*) of dimension m is a Hausdorf space with a differentiable structure of dimension m. If M is a manifold, a *local chart* on M (or a *local coordinate system* on M) is by definition a pair $(U_\alpha, \varphi_\alpha)$ where $\alpha \in A$. If $p \in U_\alpha$ and $\varphi_\alpha(p) = (x_1(p), ..., x_m(p))$, the set U_α is called a *coordinate neighborhood* of p and the numbers $x_i(p)$ are called *local coordinates* of p. The mapping $\varphi_\alpha : q \to (x_1(q), ..., x_m(q))$, $q \in U_\alpha$, is often denoted $\{x_1, ..., x_m\}$.

Remark 1. Condition (M_3) will often be cumbersome to check in specific instances. It is therefore important to note that the condition (M_3) is not essential in the definition of a manifold. In fact, if only (M_1) and (M_2) are satisfied, the family $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ can be extended in a unique way to a larger family $\mathfrak{M}$ of open charts such that (M_1), (M_2), and (M_3) are all fulfilled. This is easily seen by defining $\mathfrak{M}$ as the set of all open charts (V, φ) on M satisfying: (1) $\varphi(V)$ is an open set in $\boldsymbol{R}^m$; (2) for each $\alpha \in A$, $\varphi_\alpha \circ \varphi^{-1}$ is a diffeomorphism of $\varphi(V \cap U_\alpha)$ onto $\varphi_\alpha(V \cap U_\alpha)$.

Remark 2. If we let $\boldsymbol{R}^m$ mean a single point for $m = 0$, the preceding definition applies. The manifolds of dimension 0 are then the discrete topological spaces.

Remark 3. A manifold is connected if and only if it is pathwise connected. The proof is left to the reader.

An *analytic structure* of dimension m is defined in a similar fashion. In (M_2) we just replace "differentiable" by "analytic." In this case M is called an *analytic manifold.*

In order to define a *complex manifold* of dimension m we replace $\boldsymbol{R}^m$ in the definition of differentiable manifold by the m-dimensional complex space $\boldsymbol{C}^m$. The condition (M_2) is replaced by the condition that the m coordinates of $\varphi_\beta \circ \varphi_\alpha^{-1}(p)$ should be holomorphic functions of the coordinates of p. Here a function $f(z_1, ..., z_m)$ of m complex variables is called *holomorphic* if at each point $(z_1^0, ..., z_m^0)$ there exists a power series

$$\sum a_{n_1 \ldots n_m} (z_1 - z_1^0)^{n_1} \ldots (z_m - z_m^0)^{n_m},$$

which converges absolutely to $f(z_1, ..., z_m)$ in a neighborhood of the point.

The manifolds dealt with in the later chapters of this book (mostly

Lie groups and their coset spaces) are analytic manifolds. From Remark 1 it is clear that we can always regard an analytic manifold as a differentiable manifold. It is often convenient to do so because, as pointed out before for R^m, the class of differentiable functions is much richer than the class of analytic functions.

Let f be a real-valued function on a C^∞ manifold M. The function f is called *differentiable* at a point $p \in M$ if there exists a local chart $(U_\alpha, \varphi_\alpha)$ with $p \in U_\alpha$ such that the composite function $f \circ \varphi_\alpha^{-1}$ is a differentiable function on $\varphi_\alpha(U_\alpha)$. The function f is called *differentiable* if it is differentiable at each point $p \in M$. If M is analytic, the function f is said to be *analytic* at $p \in M$ if there exists a local chart $(U_\alpha, \varphi_\alpha)$ with $p \in U_\alpha$ such that $f \circ \varphi_\alpha^{-1}$ is an analytic function on the set $\varphi_\alpha(U_\alpha)$.

Let M be a differentiable manifold of dimension m and let $\mathfrak{F}$ denote the set of all differentiable functions on M. The set $\mathfrak{F}$ has the following properties:

($\mathfrak{F}_1$) Let $\varphi_1, \ldots, \varphi_r \in \mathfrak{F}$ and let u be a differentiable function on R^r. Then $u(\varphi_1, \ldots, \varphi_r) \in \mathfrak{F}$.

($\mathfrak{F}_2$) Let f be a real function on M such that for each $p \in M$ there exists a function $g \in \mathfrak{F}$ which coincides with f in some neighborhood of p. Then $f \in \mathfrak{F}$.

($\mathfrak{F}_3$) For each $p \in M$ there exist m functions $\varphi_1, \ldots, \varphi_m \in \mathfrak{F}$ and an open neighborhood U of p such that the mapping $q \to (\varphi_1(q), \ldots, \varphi_m(q))$ $(q \in U)$ is a homeomorphism of U onto an open subset of R^m. The set U and the functions $\varphi_1, \ldots, \varphi_m$ can be chosen in such a way that each $f \in \mathfrak{F}$ coincides on U with a function of the form $u(\varphi_1, \ldots, \varphi_m)$ where u is a differentiable function on R^m.

The properties ($\mathfrak{F}_1$) and ($\mathfrak{F}_2$) are obvious. To establish ($\mathfrak{F}_3$) we pick a local chart $(U_\alpha, \varphi_\alpha)$ such that $p \in U_\alpha$ and write $\varphi_\alpha(q) = (x_1(q), \ldots, x_m(q)) \in R^m$ for $q \in U_\alpha$. Let S be a compact neighborhood of $\varphi_\alpha(p)$ in R^m such that S is contained in the open set $\varphi_\alpha(U_\alpha)$. Then as shown earlier, there exists a differentiable function ψ on R^m such that ψ has compact support† contained in $\varphi_\alpha(U_\alpha)$ and such that $\psi(s) = 1$ for all $s \in S$. Let $U = \varphi_\alpha^{-1}(\mathring{S})$ where $\mathring{S}$ is the interior of S and define the function φ_i $(1 \leqslant i \leqslant m)$ on M by

$$\varphi_i(q) = \begin{cases} 0 & \text{if } q \notin U_\alpha, \\ x_i(q)\,\psi(\varphi_\alpha(q)) & \text{if } q \in U_\alpha. \end{cases}$$

Then the set U and the functions $\varphi_1, \ldots, \varphi_m$ have the property stated in ($\mathfrak{F}_3$). In fact, if $f \in \mathfrak{F}$, then the function $f \circ \varphi_\alpha^{-1}$ is differentiable on the set $\varphi_\alpha(U_\alpha)$.

† The *support* of a function is the closure of the set where the function is different from 0.

Proposition 1.1. *Suppose M is a Hausdorf space and m an integer > 0. Assume $\mathfrak{F}$ is a collection of real-valued functions on M with the properties $\mathfrak{F}_1$, $\mathfrak{F}_2$, and $\mathfrak{F}_3$. Then there exists a unique collection $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ of open charts on M such that (M_1), (M_2), and (M_3) are satisfied and such that the differentiable functions on the resulting manifold are precisely the members of $\mathfrak{F}$.*

For the proof we select for each $p \in M$ the functions $\varphi_1, \ldots, \varphi_m$ and the neighborhood U of p given by $\mathfrak{F}_3$. Putting $U_\alpha = U$ and $\varphi_\alpha(q) = (\varphi_1(q), \ldots, \varphi_m(q))$ $(q \in U)$ we obtain a collection $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ of open charts on M satisfying (M_1). The condition (M_2) is also satisfied in virtue of $\mathfrak{F}_3$. As remarked earlier, the collection $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ can then be extended to a collection $(U_\alpha, \varphi_\alpha)_{\alpha\in A^*}$ which satisfies (M_1), (M_2), and (M_3). This induces a differentiable structure on M and each $g \in \mathfrak{F}$ is obviously a differentiable function. On the other hand, suppose that f is a differentiable function on the constructed manifold. If $p \in M$, there exists a local chart $(U_\alpha, \varphi_\alpha)$ where $\alpha \in A^*$ such that $p \in U_\alpha$ and such that $f \circ \varphi_\alpha^{-1}$ is a differentiable function on an open neighborhood of $\varphi_\alpha(p)$. Owing to (M_2) we may assume that $\alpha \in A$. There exists a differentiable function u on $\mathbf{R}^m$ such that $f \circ \varphi_\alpha^{-1}(x) = u(x_1, \ldots, x_m)$ for all points $x = (x_1, \ldots, x_m)$ in some open neighborhood of $\varphi_\alpha(p)$. This means (in terms of the φ_i above) that

$$f = u(\varphi_1, \ldots, \varphi_m)$$

in some neighborhood of p. Since $p \in M$ is arbitrary we conclude from $\mathfrak{F}_2$ and $\mathfrak{F}_1$ that $f \in \mathfrak{F}$. Finally, let $(V_\beta, \psi_\beta)_{\beta\in B}$ be another collection of open charts satisfying (M_1), (M_2), and (M_3) and giving rise to the same $\mathfrak{F}$. Writing $f \circ \varphi_\alpha^{-1} = f \circ \psi_\beta^{-1} \circ (\psi_\beta \circ \varphi_\alpha^{-1})$ for $f \in \mathfrak{F}$ we see that $\psi_\beta \circ \varphi_\alpha^{-1}$ is differentiable on $\varphi_\alpha(U_\alpha \cap V_\beta)$, so by the maximality (M_3), $(U_\alpha, \varphi_\alpha) \in (V_\beta, \psi_\beta)_{\beta\in B}$ and the uniqueness follows.

We shall often write $C^\infty(M)$ instead of $\mathfrak{F}$ and will sometimes denote by $C^\infty(p)$ the set of functions on M which are differentiable at p. The set $C^\infty(M)$ is an algebra over $\mathbf{R}$, the operations being

$$(\lambda f)(p) = \lambda f(p),$$

$$(f + g)(p) = f(p) + g(p),$$

$$(fg)(p) = f(p)\, g(p)$$

for $\lambda \in \mathbf{R}$, $p \in M$, $f, g \in C^\infty(M)$.

Lemma 1.2. *Let C be a compact subset of a manifold M and let V be an open subset of M containing C. Then there exists a function $\psi \in C^\infty(M)$ which is identically 1 on C, identically 0 outside V.*

This lemma has already been established in the case $M = \boldsymbol{R}^m$. We shall now show that the general case presents no additional difficulties.

Let $(U_\alpha, \varphi_\alpha)$ be a local chart on M and S a compact subset of U_α. There exists a differentiable function f on $\varphi_\alpha(U_\alpha)$ such that f is identically 1 on $\varphi_\alpha(S)$ and has compact support contained in $\varphi_\alpha(U_\alpha)$. The function F on M given by

$$F(q) = \begin{cases} f(\varphi_\alpha(q)) & \text{if } q \in U_\alpha, \\ 0 & \text{otherwise} \end{cases}$$

is a differentiable function on M which is identically 1 on S and identically 0 outside U_α. Since C is compact and V open, there exist finitely many coordinate neighborhoods $U_1, \ldots, U_n$ and compact sets $S_1, \ldots, S_n$ such that

$$C \subset \bigcup_1^n S_i, \qquad S_i \subset U_i$$

$$\left(\bigcup_1^n U_i\right) \subset V.$$

As shown previously, there exists a function $F_i \in C^\infty(M)$ which is identically 1 on S_i and identically 0 outside U_i. The function

$$\psi = 1 - (1 - F_1)(1 - F_2) \ldots (1 - F_n)$$

belongs to $C^\infty(M)$, is identically 1 on C and identically 0 outside V.

Let M be a C^∞ manifold and $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ a collection satisfying (M_1), (M_2), and (M_3). If U is an open subset of M, U can be given a differentiable structure by means of the open charts $(V_\alpha, \psi_\alpha)_{\alpha \in A}$ where $V_\alpha = U \cap U_\alpha$ and ψ_α is the restriction of φ_α to V_α. With this structure, U is called an *open submanifold* of M. In particular, since M is locally connected, each connected component of M is an open submanifold of M.

Let M and N be two manifolds of dimension m and n, respectively. Let $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ and $(V_\beta, \psi_\beta)_{\beta \in B}$ be collections of open charts on M and N, respectively, such that the conditions (M_1), (M_2), and (M_3) are satisfied. For $\alpha \in A$, $\beta \in B$, let $\varphi_\alpha \times \psi_\beta$ denote the mapping $(p, q) \to (\varphi_\alpha(p), \psi_\beta(q))$ of the product set $U_\alpha \times V_\beta$ into $\boldsymbol{R}^{m+n}$. Then the collection $(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)_{\alpha \in A, \beta \in B}$ of open charts on the product space $M \times N$ satisfies (M_1) and (M_2) so by Remark 1, $M \times N$ can be turned into a manifold the *product* of M and N.

An immediate consequence of Lemma 1.2 is the following fact which will often be used: Let V be an open submanifold of M, f a function in $C^\infty(V)$, and p a point in V. Then there exists a function $\tilde{f} \in C^\infty(M)$ and an open neighborhood N, $p \in N \subset V$ such that f and $\tilde{f}$ agree on N.

Definition. Let M be a topological space. A *covering* of M is a collection of open subsets of M whose union if M. A covering $\{U_\alpha\}_{\alpha\in A}$ of M is said to be *locally finite* if each $p \in M$ has a neighborhood which intersects only finitely many of the sets U_α.

Definition. A Hausdorf space M is called *paracompact* if for each covering $\{U_\alpha\}_{\alpha\in A}$ of M there exists a locally finite covering $\{V_\beta\}_{\beta\in B}$ which is a refinement of $\{U_\alpha\}_{\alpha\in A}$ (that is, each V_β is contained in some U_α).

Definition. A topological space is called *normal* if for any two disjoint closed subsets A and B there exist disjoint open subsets U and V such that $A \subset U$, $B \subset V$.

It is known that a locally compact Hausdorf space which has a countable base is paracompact and every paracompact space is normal (see e.g., Kelley [1]; the notion of paracompactness is due to J. Dieudonné).

Theorem 1.3 (partition of unity). *Let M be a normal manifold and $\{U_\alpha\}_{\alpha\in A}$ a locally finite covering of M. Assume that each $\bar{U}_\alpha$ is compact. Then there exists a system $\{\varphi_\alpha\}_{\alpha\in A}$ of differentiable functions on M such that*

(i) *Each φ_α has compact support contained in U_α.*
(ii) $\varphi_\alpha \geqslant 0$, $\sum_{\alpha\in A} \varphi_\alpha = 1$.

We shall make use of the following fact (see, e.g., Kelley [1], p. 171):

Let $\{U_\alpha\}_{\alpha\in A}$ be a locally finite covering of a normal space M. Then each set U_α can be shrunk to a set V_α, such that $\bar{V}_\alpha \subset U_\alpha$ and $\{V_\alpha\}_{\alpha\in A}$ is still a covering of M.

To prove Theorem 1.3 we first shrink the U_α as indicated and thus get a new covering $\{V_\alpha\}_{\alpha\in A}$. Owing to Lemma 1.2 there exists a function $\psi_\alpha \in C^\infty(M)$ of compact support contained in U_α such that ψ_α is identically 1 on V_α and $\psi_\alpha \geqslant 0$ on M. Owing to the local finiteness the sum $\sum_{\alpha\in A} \psi_\alpha = \psi$ exists. Moreover, $\psi \in C^\infty(M)$ and $\psi(p) > 0$ for each $p \in M$. The functions $\varphi_\alpha = \psi_\alpha/\psi$ have the desired properties (i) and (ii).

The system $\{\varphi_\alpha\}_{\alpha\in A}$ is called a partition of unity *subordinate to the covering* $\{U_\alpha\}_{\alpha\in A}$.

§ 2. Tensor Fields

1. Vector Fields and 1-Forms

Let A be an algebra over a field K. A *derivation* of A is a mapping $D: A \to A$ such that

(i) $D(\alpha f + \beta g) = \alpha Df + \beta Dg$ for $\alpha, \beta \in K$, $f, g \in A$;
(ii) $D(fg) = f(Dg) + (Df)g$ for $f, g \in A$.