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A. L. Onishchik　E. B. Vinberg (Eds.)

Lie Groups and Lie Algebras II

Discrete Subgroups of Lie Groups and Cohomologies of Lie Groups and Lie Algebras

李群与李代数II

李群的离散子群，李群与李代数的上同调

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

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E. B. Vinberg, V. V. Gorbatsevich
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Chapter 0
Introduction

The foundations of the theory of discrete subgroups of Lie groups were laid down in the fifties and sixties of this century in the papers of A. I. Mal'tsev, G. Mostow, L. Auslander, and a number of other mathematicians. The way had been prepared by other investigations into special classes of discrete groups, owing their origins to arithmetic, geometry, the theory of functions, and to physics.

The first nontrivial discrete subgroup—the subgroup $SL_2(\mathbb{Z})$ of the group $SL_2(\mathbb{R})$, later called the modular Klein group—was considered in essence by Lagrange and Gauss in their investigations into the arithmetic of quadratic forms in two variables. Its natural generalization was the subgroup $SL_n(\mathbb{Z})$ of the group $SL_n(\mathbb{R})$. The investigation of this group, as a discrete group of transformations of the space of positive definite quadratic forms in n variables, constituted the objective of reduction theory, worked out by A. N. Korkin and E. I. Zolotarev, Hermite, Minkowski, and others in the second half ot the nineteenth century, and at the beginning of this.

A number of other arithmetically defined discrete subgroups of the classical Lie groups—the groups of units of rational quadratic forms, the groups of units of simple algebras over the field $\mathbb{Q}$ of rationals, the group of integer symplectic matrices—were studied in the first half of this century by B. A. Venkov, H. Weyl, C. L. Siegel and others.

In the theory of functions of a complex variable the integration of algebraic functions, and, more generally, the solution of linear differential equations with algebraic coefficients, led to the consideration of certain special functions, later called automorphic, invariant relative to various discrete subgroups of the group $SL_2(\mathbb{R})$, operating in the upper halfplane by fractional-linear transformations. Some of the discrete subgroups of the group $SL_2(\mathbb{R})$ arising in this way were studied in the middle of the nineteenth century by Hermite, Dedekind, and Fuchs. Among these was the group $SL_2(\mathbb{Z})$, but represented in a form different from that of Lagrange and Gauss. A wide class of such groups, including the group $SL_2(\mathbb{Z})$ and some subgroups of $SL_2(\mathbb{R})$ commensurable with it, were studied by Klein. Almost simultaneously, Poincaré in 1881–1882 gave a geometrical description of all the discrete groups of fractional-linear transformations of the upper halfplane, called by him the Fuchsian groups.

In the first half of this century a number of separate classes of meromorphic functions of several variables were considered. These functions were connected with arithmetically defined discrete subgroups of the groups $(SL_2(\mathbb{R}))^k$ (the modular functions of Hilbert), $\mathrm{Sp}_{2n}(\mathbb{R})$ (the modular functions of Siegel), and of other semisimple Lie groups.

In crystallography, beginning at the end of the last century, symmetry groups of crystal structures were studied. These are discrete subgroups of the group of motions of three-dimensional Euclidean space. E. S. Fedorov and A. Schoenflies obtained the classification of such groups. Analogous groups of motions of n-dimensional Euclidean space were studied in 1911 by Bieberbach.

Another branch was the study of discrete subgroups of solvable Lie groups, in particular abelian and nilpotent. The first result on such groups, equivalent to the description of discrete subgroups in $\mathbb{R}^2$, was obtained by Jacobi in the first half of the last century, in the course of describing the periods of meromorphic functions.

In the present work we have tried to systematize all the basic results on the theory of discrete subgroups of Lie groups. Most of it has the character of a survey. But in those cases when there is a short proof, and in particular when no short proof has yet been published, we present one. Apart from the original papers, our basic sources were: the monographs of Raghunathan (1972), Mostow (1973) and Zimmer (1984), the surveys of H.-C. Wang (1972), Mostow (1978A), Auslander (1973), Margulis (1974), and finally the notes for specialized courses given by the first-named author at Moscow State University.

A more detailed exposition of the theory of discrete subgroups of motions in spaces of constant curvature is given in the paper of Vinberg and Shvartsman (1988) in volume 29 of this Encyclopaedia, which deals particularly with spaces of constant curvature.

We have adopted the following notations and conventions. $\mathbb{N}$ denotes the natural numbers, $\mathbb{Z}$ the integer numbers, $\mathbb{Q}$ the field of the rationals, $\mathbb{R}$ that of the reals, and $\mathbb{C}$ that of the complex numbers.

If a Lie group is denoted by a capital latin letter, such as H, then its tangent Lie algebra will be denoted by the corresponding small gothic letter, in the above case $\mathfrak{h}$.

A connected component of a topological group G will be denoted by G^0. The universal covering of G will be denoted by $\tilde{G}$. If H is a subset of G, we will denote by $\overline{H}$ its closure in the topology of the group G.

If H is a subset of the affine manifold X, we will denote by ${}^aH$ its closure in the Zariski topology of that manifold.

$N_G(H)$ is the normalizer of the subgroup H in the group G, $Z(G)$ is the center of the group G, and $Z_G(a)$ is the centralizer of the element $a \in G$ in the group G.

We will denote by $A \rtimes B$, or, more precisely, by $A \underset{\varphi}{\rtimes} B$, where $\varphi : A \to \operatorname{Aut} B$, the semidirect product of the groups A and B.

$\mathbb{O}_n$ is the orthogonal group, $\mathbb{U}_n$ is the unitary group, $\mathbb{O}_{n,1}$ is the pseudo-orthogonal group, and Sp_{2n} is the symplectic group.

We will say a few words about the use of the terms "algebraic manifold" and "algebraic group" in this paper.

Throughout, unless specified otherwise, we understand by *algebraic manifold (group)* a real algebraic manifold (group), i.e. an algebraic manifold (group) defined over $\mathbb{R}$. It is identified with the set (group) of its real points, which by definition is assumed to be dense in the sense of Zariski in the set of complex points. Along with this we consider, mainly in §6 of Chap. 3, complex algebraic manifolds (groups), identified with the sets (groups) of their complex points.

The expression "algebraic k-manifold (k-group)" means "real algebraic manifold (group), defined over a subfield $k \in \mathbb{R}$". We understand similarly the expression "complex algebraic k-manifold (k-group)", where k is any numerical field.

If X (respectively G) is a real or complex algebraic k-manifold (respectively k-group), then for any field $K \supset k$ we denote by $X(K)$ (respectively $G(K)$) the set (respectively group) of K-points of the manifold K (respectively group G).

In conclusion we would like to thank A. N. Starkov, who drew our attention to a number of inaccuracies in the Russian original, making it possible to make the appropriate corrections in the English translation.

Chapter 1
Discrete Subgroups of Locally Compact Topological Groups

Throughout this chapter, when the term "locally compact group" is encountered, we have in mind a locally compact topological group with a countable basis of open sets.

§1. The Simplest Properties of Lattices

1.1. Definition of a Discrete Subgroup. Examples. A subgroup Γ of a topological group G is said to be *discrete,* if Γ is a discrete subset of the topological group G. This is equivalent to the existence in the group G of a neighborhood $U(e)$ of the unit element e such that $\Gamma \cap U(e) = \{e\}$.

Examples 1.1. The following examples are discrete subgroups:
a) The subgroup of integers in the additive group $\mathbb{R}$ of real numbers;
b) The integer linear span $\mathbb{Z}e_1 + \ldots + \mathbb{Z}e_m$ of a linearly independent family $e_1, \ldots, e_m$ of vectors in an n-dimensional real vector space V;
c) The additive subgroup of any algebraic number field k naturally embedded in the adele group A_k (see Weil 1982);
d) The subgroup $GL_n(\mathbb{Z})$ in the group $GL_n(\mathbb{R})$;
e) A finite subgroup in any topological group.

We note that any discrete subgroup of a compact group is finite.

We recall that on each locally compact group G there exists a right-invariant Borel measure, unique up to a factor. This measure is called a *right-invariant Haar measure on the group* G (see Kirillov 1978).

Fix a right-invariant Haar measure μ on the group G. Since left and right shifts by the elements of the group G commute, then a left shift $l_g(\mu)$ of the measure μ is once again a right-invariant measure. Therefore $l_g(\mu) = \chi(g)\mu$, where the function $\chi(g)$ is a character of the group G.

A group G is said to be *unimodular* if $\chi(g) \equiv 1$. This means that a right-invariant Haar measure is also left-invariant.

Now suppose that Γ is a discrete subgroup of a locally compact group G. Then a right-invariant Haar measure μ on G induces a measure on the quotient space G/Γ, which we will denote by $\overline{\mu}$.

A discrete subgroup Γ of a locally compact group G is said to be a *lattice* if the volume of the quotient space G/Γ, relative to the measure $\overline{\mu}$, is finite. In what follows we will denote that measure by $v(G/\Gamma)$, and call it the *covolume* of the lattice Γ.

If the quotient space G/Γ is compact, then a lattice Γ is said to be *uniform*. One also says that Γ is a uniform discrete subgroup of the group G.

Examples 1.2.

a) The discrete subgroup Γ of Example 1.1b) is a lattice in V if and only if $m = n$. In this case $\Gamma = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$ is a uniform lattice, and the quotient space V/Γ is an n-dimensional torus. For details, see Chap.2, 1.2.

b) The subgroup $SL_2(\mathbb{Z})$ is a lattice in the group $SL_2(\mathbb{R})$. However the quotient space $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ is non-compact for $n \geq 2$ (see Chap. 3, 2.3).

c) The discrete subgroup k of Example 1.1c) is a uniform lattice in $\mathbf{A}_k$ (Weil 1982).

We will consider one necessary condition for the existence of a lattice.

Proposition 1.3. *If a locally compact group G contains a lattice Γ, then G is unimodular.*

◀ Indeed,

$$\overline{\mu}(G/\Gamma) = \overline{\mu}(g^{-1}(G/\Gamma)) = \overline{l_g(\mu)}(G/\Gamma) = \chi(g)\overline{\mu}(G/\Gamma) ,$$

from which it follows that $\chi(g) \equiv 1$. ▶

As is clear from the following example, the necessary condition just presented is not sufficient for the existence of a lattice in a locally compact group.

Example 1.4. Suppose that $G = \mathbb{Q}_p^+$ is the additive group of p-adic numbers. Since $\lim_{n\to\infty} p^n a = 0$ for any $a \in \mathbb{Q}_p^+$, there do not exist any nontrivial discrete subgroups. On the other hand, the group G is abelian and therefore unimodular.

Example 1.5. Suppose that $G = \mathrm{Aff}\,\mathbb{R}^1$ is the group of affine transformations of the line. It is isomorphic to the matrix group

$$\left\{ g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \,\middle|\, a, b \in \mathbb{R},\ a \neq 0 \right\}.$$

The measure $dg = \frac{da\,db}{a}$ is a right-invariant Haar measure on the group G, as is easily verified, but it is not left-invariant. Therefore the group $\mathrm{Aff}\,\mathbb{R}^1$ is not unimodular. That means that it cannot contain lattices, although it does contain nontrivial discrete subgroups, for example the subgroup of matrices of the form

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \,\middle|\, b \in \mathbb{Z} \right\}.$$

Proposition 1.6 (Garland and Goto 1966). *If a connected Lie group G contains a lattice, then the group of inner automorphisms of the group G is closed in the group of all of its automorphisms.*

We note that for a Lie group of general type, so far no simple sufficient condition has been found for the existence in it of a lattice (see Chap. 4, 1.1).

Suppose that G is a locally compact group, and that Γ is a lattice in it. We will denote by π the canonical mapping $G \to G/\Gamma$.

Theorem 1.7 (Raghunathan 1972). *For an arbitrary sequence $\{g_n\}$ of elements of the group G, the sequence $\{\pi(g_n)\}$ is discrete if and only if there exists in Γ a sequence $\{\gamma_n\}$ such that*

a) $\gamma_n \neq e$;
b) $g_n \gamma_n g_n^{-1} \to e$ as $n \to \infty$.

◄ Choose in G an increasing family of compacta $K_1 \subset K_1 \subset \cdots \subset K_n \subset \cdots$, such that $G = \cup_1^\infty K_n$. In view of the finiteness of the volume of the quotient space G/Γ, the sequence $\varepsilon_n = v(G/\Gamma - \pi(K_n))$ tends to zero. We choose in G a fundamental system of compact neighborhoods V_n of the identity, such that $v(V_n) > \varepsilon_n$, and put $U_n = V_n^{-1} V_n$.

Suppose that $\pi(g_n)$ does not have a limit point. Since the set $\pi(U_n K_n)$ is compact, then $\pi(g_N) \notin \pi(U_n K_n)$ for almost all N. Hence it easily follows that $\pi(V_n g_N) \cap \pi(V_n K_n) = \varnothing$ for almost all N. Further, it is obvious that

$$v(V_n g_N) = v(V_N) > \varepsilon_n > v(G/\Gamma - \pi(V_n K_n)).$$

Since the sets $\pi(V_n g_N)$ and $\pi(V_n K_n)$ do not intersect, then, for almost all N, the set $V_n g_N$ cannot be mapped in a 1–1 way onto G/Γ.

Accordingly, for almost all N there exists an element $\gamma_N \in \Gamma$, $\gamma_N \neq e$, such that $vg_N = v'g_N\gamma_N$ for certain elements $v, v' \in V_n$. i.e. $g_N\gamma_N g_N^{-1} \in U_n$. The first part of the proposition is proved.

Now suppose that the sequence $g_n \in G$ is such that one can choose elements $\gamma_n \in \Gamma$, $\gamma_n \neq e$, for which the sequence $g_n\gamma_n g_n^{-1} \to e$ as $n \to \infty$, and also such that the sequence $\pi(g_n)$ has a limit point $\pi(g)$ in the space G/Γ. Passing if necessary to a subsequence, we may suppose that there exist $\alpha_n \in \Gamma$ such that $\lim_{n\to\infty} g_n\alpha_n = g$ in the group G. By hypothesis, $\lim_{n\to\infty} g_n\gamma_n g_n^{-1} = \lim_{n\to\infty}(g_n\alpha_n)(\alpha_n^{-1}\gamma_n\alpha_n)(\alpha_n^{-1}g_n^{-1}) = e$, and since $\lim_{n\to\infty} g_n\alpha_n = g$, then $\lim_{n\to\infty} \alpha_n^{-1}\gamma_n\alpha_n = e$. Since the group G is discrete, then for almost all n, $\alpha_n^{-1}\gamma_n\alpha_n = e$, i.e. $\gamma_n = e$ for almost all n. This is a contradiction. ▶

Remark. In the proof of the second part of Theorem 1.7, we used only the discreteness of the group Γ.

If G is a locally compact abelian group, then every lattice Γ in the group G is uniform. This is an immediate consequence of Theorem 1.7 (see also Corollary 1.2 of Chap. 2).

Proposition 1.8 (S. P. Wang 1976b). *Suppose that G is a locally compact group, Γ a lattice in G, and X a subset in G. Then the two following conditions are equivalent:*

a) The set $\pi(X)$ is relatively compact in G/Γ;
b) For any compact neighborhood K of the point e in the group G, and for any $x \in X$, the number of elements in the intersection $xKx^{-1} \cap \Gamma$ does not exceed some constant, depending only on K.

We conclude this subsection with the formulation of a property of uniform lattices, essentially proved in (Selberg 1960).

For any element $g \in G$ and any subgroup $\Gamma \subset G$, we will denote the set $\{\gamma g\gamma^{-1}, \gamma \in \Gamma\}$ by $C(\Gamma, g)$.

Proposition 1.9. *If Γ is a uniform lattice in G and the set $C(\Gamma, g)$ is discrete, then the set $C(G, g)$ is closed.*

1.2. Commensurability and Reducibility of Lattices. Many interesting properties of discrete subgroups in topological groups are properties of classes of commensurable subgroups.

Two subgroups Γ and Γ' in a group are said to be *commensurable* if $|\Gamma : \Gamma \cap \Gamma'| < \infty$ and $|\Gamma' : \Gamma \cap \Gamma'| < \infty$.

Commensurability is an equivalence relation on the set of subgroups of the group G. We will denote it by the symbol "$\sim$".

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