An Introduction to FRACTIONAL CALCULUS

A.M. Mathai ▪ H.J. Haubold

Mathematics Research Developments

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A.M. Mathai is an Emeritus Professor of Mathematics and Statistics at McGill University in Canada, and Director of the Centre for Mathematical and Statistical Sciences in India. He has published over 300 research papers and more than 25 books on topics in mathematics, statistics, physics, astrophysics, chemistry, and biology. He is a Fellow of the Institute of Mathematical Statistics and the National Academy of Sciences of India, served as President of the Mathematical Society of India, and is a member of the International Statistical Institute. Since 1974 Mathai has led an international cooperation of mathematicians and natural scientists in a research program that focuses on information theory and entropic functional, distributions as emerging from random processes in statistics and physics, and the statistical characterization of random variables and dynamics in terms of fractional calculus and generalized hypergeometric functions.

H.J. Haubold is a Professor of Theoretical Astrophysics. He has published more than 200 research papers and more than 10 books on topics of physics, astrophysics and the development of basic space science worldwide. In cooperation with A.M. Mathai, he was one of the initiators of the United Nations Basic Space Science Initiative (UN BSSI) for the worldwide development of astronomy, physics, and mathematics that was implemented in 1991-2015. He has contributed to A.M. Mathai's research program on statistics and probability for physics since the 1970s focusing entropy, probability, and dynamics. This research pursues, among other themes, the interpretation of solar neutrino detection in terms of new physics by utilizing fractional calculus.

This is a modified version of Module 10 of the Centre for Mathematical and Statistical Sciences (CMSS). CMSS modules are notes prepared on various topics with many examples from real-life situations and exercises so that the subject matter becomes interesting to students. These modules are used for undergraduate level courses and graduate level training in various topics at CMSS. Aside from Module 8, these modules were developed by Dr. A. M. Mathai, Director of CMSS and Emeritus Professor of Mathematics and Statistics, McGill University, Canada. Module 8 is based on the lecture notes of Professor W. J. Anderson of McGill University, developed for his undergraduate course (Mathematics 447). Professor Dr. Hans J. Haubold has been a research collaborator of Dr. A.M. Mathai's since 1984, mainly in the areas of astrophysics, special functions and statistical distribution theory. He is also a lifetime member of CMSS and a Professor at CMSS. A large number of papers have been published jointly in these areas since 1984. These CMSS modules are printed at CMSS Press and published by CMSS. Copies are made available to students free of charge, and to researchers and others at production cost. For the preparation of the initial drafts of all these modules, financial assistance was made available from the Department of Science and Technology, the Government of India (DST), New Delhi under project number SR/S4/MS:287/05. Hence, the authors would like to express their thanks and gratitude to DST, the Government of India, for its financial assistance.

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Preface

This is a modified version of CMSS (Centre for Mathematical and Statistical Sciences) Module 10. CMSS Modules are notes prepared on various topics with lots of examples from real-life situations so that the subject-matter becomes interesting to students, and with lots of exercises for the students to do. These Modules are used for undergraduate level and graduate level training in various topics at CMSS. These Modules are developed by Dr A.M. Mathai, Director of CMSS and Emeritus Professor of Mathematics and Statistics, McGill University, Canada, except Module 8 on Basic Stochastic Processes. Module 8 is based on the lecture notes of Professor W.J. Anderson of McGill University, developed for his undergraduate course (Mathematics 447) and hence Module 8 is brought out in the name of W.J. Anderson.

Professor Dr. Hans J. Haubold is a research collaborator of Professor Dr. A.M. Mathai from 1984 onward, mainly in the areas of astrophysics, special functions and statistical distribution theory. He is also a Life Member of CMSS and a Professor in CMSS. A large number of papers are published jointly in these areas since 1984. Research level monograph and books brought out jointly are the following: *Modern Problems in Nuclear and Neutrino Astrophysics* (A.M. Mathai and H.J. Haubold,1988, Akademie-Verlag, Berlin); *Special Functions for Applied Scientists* (A.M.Mathai and H.J. Haubold, 2008, Springer, New York); *The H-function: Theory and Applications* (A.M.Mathai, R.K. Saxena and H.J. Haubold, 2010, Springer, New York). The 2008 and 2010 books are very popular, citation-wise.

These CMSS Modules are printed at CMSS Press and published by CMSS. Copies ere made available to students free of cost and to researchers and others at production cost. For the preparation of the initial drafts of all these modules, financial assistance was available from the Department of Science and Technology, Government of India (DST), New Delhi under project number

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Introduction

The first draft of this book was based on the series of lectures given by Professor A.M.Matai on the topic of fractional calculus in the Department of Science and Technology, Government of India, New Delhi (DST) Centre for Interdisciplinary Mathematical Sciences (CIMS) at Banaras Hindu University (BHU). The first draft of chapters 1 to 3 were brought out by CIMS of BHU as A.M. Mathai's lecture notes on fractional calculus. Later the topic of fractional calculus was incorporated as an integral part of the sequence of the all-India research level courses conducted by CMSS, known as SERC Schools (now SERB Schools), sponsored by DST, New Delhi's SERC (Science and Engineering Research Council) Division. These sequences of Schools at CMSS started in 1995. Each School used to be of 6 weeks duration with 4 hours of lectures and 4 hours of problem-solving sessions every day Monday-Friday for 6 weeks. From 2006 onward these Schools are of 4 weeks duration only. Professor Dr. H.J. Haubold was an integral part of these Schools, one of the organizers and one of the foreign lecturers. Each School had 2 to 3 foreign lecturers and 2 to 3 lecturers from India. Then A.M. Mathai and H.J. Haubold organized a national level conference on fractional calculus in 2012 at CMSS Pala Campus. Then the lecture notes were updated and brought out as Module 10 in August 2014. A second printing of Module 10 took place in 2015. The present book is an updated version of the 2015 printing and brought out in July 2016.

The readers and users of Module 10 enjoyed the book due to many reasons. The main reasons, as stated by the users, are the following: The material is developed slowly with lots of worked examples and exercises. Analysis is kept at minimum essential levels. The material is not chocked with analysis. As a result, the users of this book claim that it is the best book for learning the subject-matter. Apart from the routine materials on Mittag-Leffler functions, fractional integrals, fractional derivatives, fractional differential equations, new topics appearing in this book for the first time are the connection of fractional calculus to statistical distribution theory, fractional integrals in the real and complex matrix-variate cases, fractional calculus in the complex domain, fractional derivatives in the matrix-variate case. Geometrical interpretations are given for the left-sided and right-sided fractional integrals. Interpretations of fractional integrals are given as fractions of total probabilities, and as densities of sum and difference of real scalar positive random variables. Fractional calculus is a jungle as far as notations are concerned. Hence unified notations are developed in this book which will make the readers to identify and locate various items at ease.

Chapter 1 gives a geometrical interpretation of fractional integrals for the real scalar variable case. Another interpretation given for fractional integrals is as fractions of total probability in beta and gamma densities. Another interpretation given is fractional integrals of the first and second kind as densities of sum and difference of independently distributed positive real scalar random variables. Geometrical interpretations are given for right-sided and left-sided integrals or the first and second kind integrals.

Chapter 2 is on Mittag-Leffler functions, generalization and properties. Gorenflo and Mainardi call Mittag-Leffler function as the queen function in fractional calculus. A new topic introduced in this chapter is a generalized Mittag-Leffler statistical density introduced by Mathai recently, along with its properties.

Chapter 3 gives the basic ideas of fractional integral, fractional derivative, their Laplace and Mellin transforms and properties. From chapter 4 onward new materials are introduced through Kober fractional integral operators interpreting them as Mellin convolutions of products and ratios thereby establishing a connection to distributions of products and ratios of positive real scalar random variables. The theory of fractional calculus is developed parallel to statistical distribution theory in this chapter. Fractional calculus of matrix-variate functions is also developed in this chapter.

Chapter 5 deals with fractional integrals and derivatives for many real scalar and many matrix variables. The materials, developed recently by the authors, appear for the first time in a book. Chapter 6 is on fractional differential equations. Apart from routine materials, some recent results are also added here. In Chapter 8 a definition for fractional derivatives in the matrix variate case is developed. The materials corresponding to the ones in chapters 7 and 8 are also developed recently by Mathai but these are not included in the book in order not to frighten away the readers.

The first draft of this book as Module 10 of CMSS was developed with financial support from DST, Government of India, New Delhi, under Project Number SR/S4/MS:287/05. The authors would like to express their sincere gratitude to DST, New Delhi, for the financial assistance.

A. M. Mathai and H. J. Haubold
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CONTENTS

List of Symbols

In the following list, R-L means "Riemann-Liouville"

General notation: all fractional integrals of order α, with $-\alpha$ denoting integrals and $+\alpha$ denoting derivatives; first kind denoted by 1, second kind by 2; scalar case by small letters and matrix case by capital letters; $D =$ for the Riemann-Liouville operators, C for the Caputo operators, W for teh Weyl operators, S for the Saigo operators. General notation is the following: