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主编 钱伟长

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王宽诚教育基金会敬赠

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谨以此书纪念本会创建人、故董事会主席王宽诚先生

王宽诚教育基金会

DEDICATED TO THE MEMORY OF MR. K.C. WONG, FOUNDER OF THE FOUNDATION AND THE LATE CHAIRMAN OF THE BOARD OF DIRECTORS

K.C. WONG EDUCATION FOUNDATION

王宽诚教育基金会简介

王宽诚先生(1907—1986)为香港著名爱国人士，热心祖国教育事业，生前为故乡宁波的教育事业做出积极贡献。1985 年独立捐巨资创建王宽诚教育基金会，其宗旨在于为国家培养高级技术人才，为祖国四个现代化效力。

王宽诚先生在世时聘请海内外著名学者担任基金会考选委员会和学务委员会委员，共商大计，确定采用“送出去”和“请进来”的方针，为国家培养各科专门人才，提高内地和港澳高等院校的教学水平，资助学术界人士互访以促进中外文化交流。在此方针指导下，1985、1986 两年，基金会在国家教委支持下，选派学生 85 名前往英、美、加拿大、德国、瑞士和澳大利亚各国攻读博士学位，并计划资助内地学者赴港澳讲学，资助港澳学者到内地讲学，资助美国学者来国内讲学。正当基金会事业初具规模、蓬勃发展之时，王宽诚先生一病不起，于 1986 年年底逝世。这是基金会的重大损失，共事同仁，无不深切怀念，不胜惋惜。

自 1987 年起，王宽诚教育基金会继承王宽诚先生为国家培养高级技术人才的遗愿，继续对中国内地、台湾及港澳学者出国攻读博士学位、博士后研究及学术交流提供资动。委请国家教育部、中国科学院和上海大学校长钱伟长教授等逐年安排资助学术交流的项目。相继与（英国）皇家学会、英国学术院、法国科研中心、德国学术交流中心等著名欧州学术机构合作，设立“王宽诚（英国）皇家学会奖学金”、“王宽诚英国学术院奖学金”、“王宽诚法国科研中心奖学金”、“王宽诚德国学术交流中心奖学金”，资助具有博士学位、副教授或同等学历职称的中国内地学者前往英国、法国、德国等地的高等学府及科研机构进行为期 3 至 12 个月之博士后研究。

王宽诚教育基金会过去和现在的工作态度一贯以王宽诚先生倡导的“公正”二字为守则，谅今后基金会亦将秉此行事，奉行不辍，借此王宽诚教育基金会《学术讲座汇编》出版之际，特简明介绍如上。王宽诚教育基金会日常工作繁忙，基金会各位董事均不辞劳累，做出积极贡献。

钱 伟 长

二〇〇五年六月

前　言

王宽诚教育基金会是由已故全国政协常委、香港著名工商企业家王宽诚先生(1907—1986)出于爱国热忱，出资一亿美元于1985年在香港注册登记创立的。

1987年，基金会开设“学术讲座”项目，此项目由当时的全国政协委员、历任第六、七、八、九届全国政协副主席、著名科学家、中国科学院院士、上海大学校长、王宽诚教育基金会贷款留学生考选委员会主任委员兼学务委员会主任委员钱伟长教授主持。由钱伟长教授亲自起草设立“学术讲座”的规定，资助内地学者前往香港、澳门讲学，资助美国学者来中国讲学，资助港澳学者前来内地讲学，用以促进中外学术交流，提高内地及港澳高等院校的教学质量。

本汇编收集的文章，均系各地学者在“学术讲座”活动中的讲稿，文章内容有科学技术，有历史文化，有经济专论，有文学，有宗教和中国古籍研究等。本汇编涉及的学术领域颇为广泛，而每篇文章都有一定的深度和广度，分期分册以《王宽诚教育基金会学术讲座汇编》的名义出版，并无偿分送国内外部分高等院校、科研机构和图书馆，以广流传。

王宽诚教育基金会除资助“学术讲座”学者进行学术交流之外，在钱伟长教授主持的项目下，还资助由国内有关高等院校推荐的学者前往欧、美、亚、澳等参加国际学术会议，出访的学者均向所出席的会议提交论文，这些论文亦颇有水平，本汇编亦将其收入，以供参考。

王宽诚教育基金会学务委员会

凡　例

(一) 编排次序

本书所收集的王宽诚教育基金会学术讲座的讲稿及由王宽诚教育基金会资助学者赴欧、美、亚、澳等参加国际学术会议的论文均按照文稿日期先后或文稿内容编排刊列，不分类别。

(二) 分期分册出版并作简明介绍

因文稿较多，为求便于携带，有利阅读与检索，故分期分册出版，每册约 150 页至 200 页不等。为便于读者查考，每篇学术讲座的讲稿均注明作者姓名、学位、职务、讲学日期、地点、访问院校名称。内地及港、澳学者到欧、美、澳及亚洲的国家和地区参加国际学术会议的论文均注明学者姓名、参加会议的名称、时间、地点和推荐的单位。上述两类文章均注明由王宽诚教育基金会资助字样。

(三) 文字种类

本书为学术性文章汇编，均以学术讲座学者之讲稿原稿或参加国际学术会议者向会议提交的论文原稿文字为准，原讲稿或论文是中文的，即以中文刊出，原讲稿或论文是外文的，仍以外文刊出。

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Nonlinear Stochastic Dynamics and Control in Hamiltonian Formulation

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Abstract The significant advances in nonlinear stochastic dynamics and control in Hamiltonian formulation mainly due to the present author and his co-workers during the past decade are reviewed. The exact stationary solutions and equivalent nonlinear system method of Gaussian-white -noises excited and dissipated Hamiltonian systems, the stochastic averaging method for quasi Hamiltonian systems, the stochastic stability, stochastic bifurcation, first-passage time and nonlinear stochastic optimal control of quasi Hamiltonian systems are summarized. Possible extension and applications of the theory are pointed out.

1. Introduction

Stochastic dynamics was originated from an effort to describe Brownian motion quantitatively a century ago. In the 1940s and 1950s, the theory of random noise, random vibration and stochastic (probabilistic) structural dynamics was developed to meet the needs in various engineering areas, such as communication, aeronautical and astronautical, mechanical, civil and ocean engineering, etc.. Since 1960s the theoretical research on stochastic dynamics has been focused on the response of nonlinear stochastic systems, stochastic stability and stochastic control. While great progress had been made until the beginning of the 1990s, the theory for multi-degree-of-freedom (MDOF) strongly nonlinear stochastic system has not been well developed [1-3].

In the past decade, the nonlinear stochastic dynamical systems were formulated as stochastically excited and dissipated Hamiltonian systems and classified into five groups based on the integrability and resonance of the associated Hamiltonian systems. An innovative theory of stochastically excited and dissipated Hamiltonian systems was proposed and developed by the present author and his co-workers. It includes the exact stationary solutions and equivalent

* 朱位秋，浙江大学机械与能源工程学院教授，中国科学院院士，由王宽诚教育基金会资助，于 2004 年 5 月前往香港科技大学、香港大学、香港理工大学讲学，此为其讲学的有关内容。

nonlinear system method of dissipated Hamiltonian systems subject to Gaussian white noise excitation, the stochastic averaging method for quasi Hamiltonian systems, the theory and approaches for the stochastic stability, stochastic bifurcation, first-passage time and nonlinear stochastic optimal control of quasi Hamiltonian systems. They constitute a Hamiltonian theoretical framework for nonlinear stochastic dynamics and control and provide a serious of procedures for solving the difficult problems in the dynamics and control of MDOF strongly nonlinear stochastic systems [4].

In the present paper, the concepts, methods and significant results of the nonlinear stochastic dynamics and control in Hamiltonian formulation are reviewed and the possible extension and applications of the theory are pointed out.

2. Hamiltonian Formulation and Classification of Nonlinear Stochastic Dynamical Systems

A controlled nonlinear stochastic dynamical system of MDOF can be described by using the following n pairs of equations :

$$\begin{aligned} \dot{Q}_i &= -\frac{\partial H}{\partial P_i} \\ \dot{P}_i &= -\frac{\partial H}{\partial Q_i} - c_{ij}\frac{\partial H}{\partial P_j} + u_i + f_{ik}\xi_k(t) \\ & i, j = 1, 2, \cdots, n; \quad k = 1, 2, \cdots, m \end{aligned} \tag{1}$$

where Q_i and P_i are generalized displacements and momenta, respectively; $H=H(Q,P)$ is Hamiltonian with continuous partial derivatives; $c_{ij} = c_{ij}(Q,P)$ denote the coefficients of quasi-linear dampings; $f_{ik} = f_{ik}(Q,P)$ denote the amplitudes of stochastic excitations; $\xi_k(t)$ are random processes, including periodic or harmonic functions in special cases; $u_i = u_i(Q,P)$ denote feedback control forces. The system governed by equation (1) is called controlled, stochastically excited and dissipated Hamiltonian system. Early Hamiltonian formulation of nonlinear stochastic systems [5] is the special case of equation (1).

The core of system (1) is the associated Hamiltonian system, which is described by Hamilton equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \ \dot{p}_i = \frac{\partial H}{\partial q_i}, \quad i = 1, 2, \cdots, n \tag{2}$$

Here, it is assumed that Hamiltonian system (2) is autonomous and characterized by Hamiltonian $H=H(\boldsymbol{q}, \boldsymbol{p})$. For mechanical/structural systems, Hamiltonian H represents the total energy (sum of kinetic energy and potential energy) of the system and it is conservative during the motion of the system. Hamiltonian systems can be classified according to the number of independent first integrals (conservative quantities or motion constants) $H_1 = H, H_2, \cdots, H_r$, which are in involution [4]. A dynamic quantity $H_i = H_i(\boldsymbol{q}, \boldsymbol{p})$ is called first integral if $[H_i, H] = 0$, and two first integrals are called in involution if $[H_i, H_j]=0$, where

$$[H_i, H_j] = \frac{\partial H_i}{\partial p_k}\frac{\partial H_j}{\partial q_k} - \frac{\partial H_i}{\partial q_k}\frac{\partial H_j}{\partial p_k}, \quad i,j=1,2,\cdots r, \ k=1,2,\cdots n \tag{3}$$

is the Poisson bracket of H_i and H_j. Hamiltonian system (2) is called non-integrable if r=1, integrable (or completely integrable) if $r = n$, and partially integrable if $1<r<n$.

For integrable Hamiltonian systems, it is possible principally to introduce the action-angle variables, I_i and θ_i, i=1,2,···, n. In terms of the action-angle variables, the Hamiltonian of an integrable Hamiltonian system is of the form H=$H(\boldsymbol{I})$ and the Hamilton equations are of the form

$$\dot{\theta}_i = \frac{\partial H(\boldsymbol{I})}{\partial I_i} = \omega_i(\boldsymbol{I}), \ \dot{I}_i = 0, \ i=1,2,\cdots,n \tag{4}$$

where $\omega_i(\boldsymbol{I})$ are the n frequencies of the Hamiltonian system. The solution to system (4) is then

$$I_i = \text{const}, \theta_i = \omega_i(\boldsymbol{I})t + \delta_i, \ i=1,2,\cdots,n \tag{5}$$

where δ_i are constants of integration. An integrable Hamiltonian system is called resonant if its frequencies satisfy at least one of the following strong resonant relations:

$$k_i^u \omega_i(\boldsymbol{I})=0, \ i=1,2,\cdots n; \ u=1,2,\cdots,\alpha \tag{6}$$

where k_i^u are integers. Otherwise, the integrable Hamiltonian system is called non-resonant.

In principle, a partially integrable Hamiltonian system can be converted into one consisting of an integrable and a non-integrable Hamiltonian subsystems by using canonical transformation, i.e., Hamiltonian system with Hamiltonian

$$H = \sum_{\eta=1}^{r-1} H_\eta(q_\eta, p_\eta) + H_r(q_r, \cdots, q_n; p_r, \cdots p_n) \tag{7-a}$$

or

$$H = \sum_{\eta=1}^{r-1} H_\eta(I_\eta) + H_r(q_r, \cdots, q_n; p_r, \cdots p_n) \tag{7-b}$$

So, a partially integrable Hamiltonian system can also be resonant or non-resonant, depending on whether the integrable Hamiltonian sub-system is resonant or not.

Thus, according to their integrability and resonance, Hamiltonian systems can be classified as five groups: non-integrable, integrable and non-resonant, integrable and resonant, partially integrable and non-resonant, and partially integrable and resonant. The behaviors of Hamiltonian systems in different groups are different. For example, the motion of an integrable and non-resonant Hamiltonian system is almost periodic and a single phase space orbit eventually cover an n-dimensional torus uniformly. The motion of a non-integrable Hamiltonian system is chaotic when its energy reaches certain value and ergodic on the $(n-1)$-dimensional energy shell.

To date, there is no general procedure to identify whether a given Hamiltonian system is integrable or not. However, there are some identification methods, such as Hamilton-Jacobi method[6], method of Lax pairs[6], Painlevé singularity analysis[7], Whittaker integrable potential[8] and Poincare map[9], each of which is applicable to some special class of Hamiltonian systems. Any single DOF autonomous Hamiltonian system and n DOF autonomous linear Hamiltonian system are integrable. Some other examples of integrable Hamiltonian systems can be found in [10,11].

Controlled, stochastically excited and dissipated Hamiltonian systems governed by equation (1) can also be classified into five groups based on the integrability and resonance of the associated Hamiltonian systems. This classification is significant since it has been shown that the functional form of the exact and approximate solutions of the systems depends on the integrablity and resonance of the associated Hamiltonian systems.

3. Exact Stationary Solution

Consider a special case of system (1), i.e., a dissipated Hamiltonian system subject to Gaussian white noise excitation. The equations of motion of the system are of the form

$$\begin{aligned} \dot{Q}_i &= \frac{\partial H'}{\partial P_i} \\ \dot{P}_i &= -\frac{\partial H'}{\partial Q_i} - c_{ij}\frac{\partial H'}{\partial P_j} + f_{ik}W_k(t) \end{aligned} \tag{8}$$

where $W_k(t)$ are Gaussian white noises in the sense of Stratonovich with correlation functions $E[W_k(t)W_l(t+\tau)] = 2D_{kl}\delta(\tau)$. Equation (8) can be rewritten as Stratonovich stochastic differential equations and then converted into Itô stochastic differential equations by adding the Wong-Zakai correction terms. These terms can be split into conservative parts and dissipative parts, which can be combined, respectively, with $-\partial H'/\partial Q_i$ and $-c_{ij}\partial H'/\partial P_j$ to form overall effective conservative forces $-\partial H/\partial Q_i$ and effective damping forces $-m_{ij}\partial H/\partial P_j$. With these accomplished, equation (8) becomes

$$\begin{aligned} \mathrm{d}\dot{Q}_i &= \frac{\partial H}{\partial P_i}\mathrm{d}t \\ \mathrm{d}\dot{P}_i &= -\left(\frac{\partial H}{\partial Q_i} + m_{ij}\frac{\partial H}{\partial P_j}\right)\mathrm{d}t + \sigma_{ik}\,\mathrm{d}B_k(t) \end{aligned} \tag{9}$$

where $H=H(\boldsymbol{Q}, \boldsymbol{P})$ and $m_{ij} = m_{ij}(\boldsymbol{Q}, \boldsymbol{P})$ are modified Hamiltonian and modified damping coefficients, respectively; $B_k(t)$ are standard wiener processes; $\sigma_{ik} = \sigma_{ik}(\boldsymbol{Q},\boldsymbol{P})$ with $\boldsymbol{\sigma\sigma}^{\mathrm{T}} = 2(\boldsymbol{fDf}^{\mathrm{T}})$. It is seen from equation (9) that $(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{P}^{\mathrm{T}})^{\mathrm{T}}$ is a vector of diffusion processes and its transition probability density is governed by a Forkker-Planck-Kolmogorov (FPK) equation. The exact transient solution to this FPK equation can generally not be obtained. So, only the exact stationary solution is considered here. The exact stationary probability density $\rho(\boldsymbol{q}, \boldsymbol{p})$ is governed by the following reduced FPK equation:

$$[\rho, H]+\frac{\partial}{\partial p_i}(m_{ij}\frac{\partial H}{\partial p_j}\rho)+\frac{1}{2}\frac{\partial^2}{\partial p_i \partial p_j}(b_{ij}\rho)=0 \tag{10a}$$

or

$$[\rho, H]+\frac{\partial}{\partial p_i}(m_{ij}\frac{\partial H}{\partial p_j}\rho)+\frac{\partial^2}{\partial p_i \partial p_j}(b_{ij}^{(i)}\rho)=0 \tag{10b}$$

where $b_{ij}=\sigma_{ik}\sigma_{jk}$ and $b_{ij}=b_{ij}^{(i)}+b_{ji}^{(j)}$. Equation (10a) or (10b) is solved subject to the boundary condition of vanishing probability flow at the boundary.

3.1 Non-integrable case

It has been shown in [12-14] that, if the associated modified Hamiltonian system with Hamiltonian H is non-integrable, then the exact stationary solution to equation (10a) or (10b) is of the form

$$\rho(\boldsymbol{q}, \boldsymbol{p})=C\exp[-\lambda(H)]|_{H=H(\boldsymbol{q},\boldsymbol{p})} \tag{11}$$

where C is a normalization constant and $\lambda(H)$ is the solution of the following set of n first-order linear ordinary differential equations

$$m_{ij}\frac{\partial H}{\partial p_j}+\frac{\partial b_{ij}^{(i)}}{\partial p_j}-b_{ij}^{(i)}\frac{\partial H}{\partial p_j}\frac{\mathrm{d}\lambda}{\mathrm{d}H}=0, i, j=1,2,\cdots,n \tag{12}$$

where $b_{ij}^{(i)}$ is replaced by $b_{ij}/2$ for equation (10a). If a consistent

$$(m_{ij}\frac{\partial H}{\partial p_j}+\frac{\partial b_{ij}^{(i)}}{\partial p_j})/b_{ij}^{(i)}\frac{\partial H}{\partial p_j}=h(H) \tag{13}$$

satisfying all the n equations in equation (12) can be found, then

$$\lambda(H)=\lambda(0)+\int_0^H h(u)\mathrm{d}u \tag{14}$$

The exact stationary solution is obtained by substituting equation (14) into equation (11). MDOF vibro-impcact systems are the examples of this kind solutions[15].

3.2 Integrable and non-resonant case

If the associated modified Hamiltonian system is integrable and non-resonant, then the exact stationary solution of equation (10a) or (10b) has the form[14]

$$\rho(\boldsymbol{q},\boldsymbol{p}) = C\exp[-\lambda(\boldsymbol{H})]\,|_{\,\boldsymbol{H}=\boldsymbol{H}(\boldsymbol{q},\boldsymbol{p})} \tag{15}$$

where $\boldsymbol{H} = [H_1 H_2 \cdots H_n]^{\mathrm{T}}$ is a n–d vector of first integrals of the associated Hamiltonian system and $\lambda(\boldsymbol{H})$ is the solution of the following set of n first-order linear partial differential equations:

$$m_{ij}\frac{\partial H}{\partial p_j} + \frac{\partial}{\partial p_j}b_{ij}^{(i)} - b_{ij}^{(i)}\frac{\partial H_s}{\partial p_j}\frac{\partial \lambda}{\partial H_s} = 0,\ i = 1,2,\cdots,n \tag{16}$$

If $\partial\lambda/\partial H_s$ can be found as functions of H_i and satisfy the following compatibility conditions:

$$\frac{\partial^2\lambda}{\partial H_{s1}\partial H_{s2}} = \frac{\partial^2\lambda}{\partial H_{s2}\partial H_{s1}}\ ,\quad s_1, s_2 = 1,2,\cdots,n \tag{17}$$

then

$$\lambda(\boldsymbol{H}) = \lambda(\boldsymbol{0}) + \int_0^{H_s}\frac{\partial\lambda}{\lambda H_s}\mathrm{d}H_s \tag{18}$$

The second term on the right hand side of equation (18) is a line integral and the integrand is a summation over s=1,2,···, n. The exact stationary solution is obtained by substituting equation (18) into equation (15). Note that equations (15)–(18) hold if $\boldsymbol{H}$ is replaced by vector $\boldsymbol{I}$ of action variables $I_1, I_2, \cdots, I_n$. Linear autonomic Hamiltonian systems subject to linear and/or nonlinear dampings and external and/or parametric excitations of Gaussian white noises are examples of this kind of solutions [4].

3.3 Integrable and resonant case

If the associated modified Hamiltonian system is integrable and resonant with α resonant relations of the form of equation (6), then the exact stationary solution is of the form

$$\rho\ (\boldsymbol{q},\ \boldsymbol{p}) = C\exp[-\lambda(\boldsymbol{I},\ \psi)]\,|_{\,\boldsymbol{I}=\boldsymbol{I}(\boldsymbol{q},\ \boldsymbol{p}),\ \psi=\psi(\boldsymbol{q},\ \boldsymbol{p})} \tag{19}$$

where $\psi=[\psi_1\ \psi_1 \cdots \psi_\alpha]$ and $\psi_u = k_i^u\theta_i$ are combinations of angle variables, and $\lambda(\boldsymbol{I},\psi)$ is the solution of the following set of first-order partial differential equations:

$$m_{ij}\frac{\partial H}{\partial p_j} + \frac{\partial b_{ij}^{(i)}}{\partial p_j} - b_{ij}^{(i)}\left(\frac{\partial I_s}{\partial p_j}\frac{\partial\lambda}{\partial I_s} + \frac{\partial\psi_u}{\partial p_j}\frac{\partial\lambda}{\partial\psi_u}\right) = 0$$
$$i, j, s = 1,2,\cdots,n;\ \ u = 1,2,\cdots,\alpha \tag{20}$$

If $\partial\lambda/\partial H_s$ and $\partial\lambda/\partial\psi_u$ can be found as functions of I_i and ψ_u and satisfy the following compatibility conditions;

$$\frac{\partial^2\lambda}{\partial I_{s1}\partial I_{s2}} = \frac{\partial^2\lambda}{\partial I_{s2}\partial I_{s1}},\ \frac{\partial^2\lambda}{\partial\psi_{u1}\partial\psi_{u2}} = \frac{\partial^2\lambda}{\partial\psi_{u2}\partial\psi_{u1}},\ \frac{\partial^2\lambda}{\partial I_s\partial\psi_u} = \frac{\partial^2\lambda}{\partial\psi_u\partial I_s}$$
$$s_1, s_2 = 1,2,\cdots,n\ \ ;\ \ u_1, u_2 = 1,2,\cdots,\alpha \tag{21}$$

then

$$\lambda(\boldsymbol{I},\psi) = \lambda(\boldsymbol{0},\boldsymbol{0}) + \int_0^{I_s}\frac{\partial\lambda}{\partial I_s}\mathrm{d}I_s + \int_0^{\psi_u}\frac{\partial\lambda}{\partial\psi_u}\mathrm{d}\psi_u \tag{22}$$

The exact stationary solution is obtained by substituting equation (22) into equation (19). Since $\boldsymbol{H}=\boldsymbol{H}(\boldsymbol{I})$, equations (19)–(20) hold if $\boldsymbol{I}$ is replaced by $\boldsymbol{H}$. Examples of this kind of solutions can be found in [4,14].

3.4 Partially integrable and non-resonant case

If the associated modified Hamiltonian system is partially integrable with Hamiltonian (7-a) or (7-b), and non-resonant, then the exact stationary solution is of the form[16]

$$\rho(\boldsymbol{q},\boldsymbol{p}) = C\exp[-\lambda(\boldsymbol{H}_1)]|_{\boldsymbol{H}_1=\boldsymbol{H}_1(\boldsymbol{q},\boldsymbol{p})} \tag{23}$$

where $\boldsymbol{H}_1 = [H_1 H_2 \cdots H_r]^{\mathrm{T}}$ and $\lambda(\boldsymbol{H}_1)$ is the solution of the following set of n first-order linear partial differential equations:

$$m_{ij}\frac{\partial H}{\partial p_j} + \frac{\partial}{\partial p_j} b_{ij}^{(i)} - b_{ij}^{(i)}\frac{\partial H_s}{\partial p_j}\frac{\partial \lambda}{\partial H_s} = 0\ ,\quad i=1,2,\cdots,n\ ;\ s=1,2,\cdots r \tag{24}$$

If $\partial\lambda/\partial H_s$ can be found as functions of H_i and satisfy the following compatibility conditions:

$$\frac{\partial^2 \lambda}{\partial H_{s1}\partial H_{s2}} = \frac{\partial^2 \lambda}{\partial H_{s2}\partial H_{s1}}\ ,\quad s_1, s_2 = 1,2,\cdots,r \tag{25}$$

then

$$\lambda(\boldsymbol{H}_1)=\lambda(\boldsymbol{0})+\int_0^{H_s}\frac{\partial\lambda}{\partial H_s}\mathrm{d}H_s\quad s=1,2,\cdots,r \tag{26}$$

The exact stationary solution is obtained by substituting equation (26) into equation (23). Equation (23)–(26) hold of $H_1, H_2, \cdots, H_r$ are replaced by $I_1, I_2, \cdots, I_{r-1}$ and H_r.

3.5 Partially integrable and resonant case

If the associated modified Hamiltonian system is partially integrable with Hamiltonian (7a) or (7b) and resonant with β resonant relations of the form of equation (6), then the exact stationary solution is of the form [16]

$$\rho(\boldsymbol{q},\ \boldsymbol{p}) = C\exp[-\lambda(I_1, I_2, \cdots, I_{r-1}, H_r, \psi_1, \cdots, \psi_\beta)] \tag{27}$$

where

$I_\eta = I_\eta(q_\eta, p_\eta)$, $H_r = H_r(q_r,\cdots,q_n,p_r,\cdots,p_n)$, $\psi_v = \psi_v(q_1,\cdots,q_{r-1},p_1,\cdots,p_{r-1})$, $\lambda(I_1,\cdots,I_{r-1},H_r,\psi_1,\cdots,\psi_\beta)$ is the solution of the following set of n first-order linear partial differential equations:

$$m_{ij}\frac{\partial H}{\partial p_j} + \frac{\partial}{\partial p_j} b_{ij}^{(i)} - b_{ij}^{(i)}\left(\frac{\partial I_\eta}{\partial p_j}\frac{\partial\lambda}{\partial I_\eta} + \frac{\partial H_r}{\partial p_j}\frac{\partial\lambda}{\partial H_r} + \frac{\partial\psi_v}{\partial p_j}\frac{\partial\lambda}{\partial\psi_v}\right) = 0,$$
$$i,j=1,2,\cdots,n\ ;\ \eta=1,2,\cdots,r-1\ ;\ v=1,2,\cdots,\beta \tag{28}$$

If $\partial\lambda/\partial I_\eta, \partial\lambda/\partial H_r$ and $\partial\lambda/\partial\psi_u$ can be obtained as functions of I_η, H_r, ψ_v and satisfy the following compatibility conditions:

$$\frac{\partial^2\lambda}{\partial I_{\eta1}\partial I_{\eta2}}=\frac{\partial^2\lambda}{\partial I_{\eta2}\partial I_{\eta1}},\frac{\partial^2\lambda}{\partial\psi_{v1}\partial\psi_{v2}}=\frac{\partial^2\lambda}{\partial\psi_{v2}\partial\psi_{v1}},\frac{\partial^2\lambda}{\partial I_\eta\partial H_r}=\frac{\partial^2\lambda}{\partial H_r\partial I_\eta},\frac{\partial^2\lambda}{\partial H_r\partial\psi_u}=\frac{\partial^2\lambda}{\partial\psi_u\partial H_r},\frac{\partial^2\lambda}{\partial I_\eta\partial\psi_u}=\frac{\partial^2\lambda}{\partial\psi_u\partial I_\eta} \tag{29}$$

then

$$\lambda(I_1,\cdots I_{r-1},H_r,\psi_1,\cdots\psi_\beta)=\lambda(0)+\int_0^{I_\eta}\frac{\partial\lambda}{\partial I_\eta}\mathrm{d}I_\eta+\int_0^{H_r}\frac{\partial\lambda}{\partial H_r}\mathrm{d}H_r+\int_0^{\psi_v}\frac{\partial\lambda}{\partial\psi_r}\mathrm{d}\psi_v \tag{30}$$

The exact stationary solution is obtained by substituting equation (30) into equation (27). Equations (27)–(30) hold if I_η are replaced by H_η. The examples of this kind of solutions can be found in [4,16].

Gyroscopic force is usually derived from generalized potential, which should be a part of Hamiltonian. The theory and method of exact stationary solution depicted above are applicable to both non-gyroscopic and gyroscopic systems[17]. They can also be applied to the following more general systems

$$\begin{aligned}&\mathrm{d}Q_i=D(\boldsymbol{Q})\frac{\partial H}{\partial P_i}\mathrm{d}t\\&\mathrm{d}P_i=-(D(\boldsymbol{Q})\frac{\partial H}{\partial Q_i}+m_{ij}(\boldsymbol{Q},\boldsymbol{P})\frac{\partial H}{\partial P_j})\mathrm{d}t+\sigma_{ik}(\boldsymbol{Q},\boldsymbol{P})\mathrm{d}B_k(t)\end{aligned} \tag{31}$$

where $D(\boldsymbol{Q})$ is any function of $\boldsymbol{Q}$. The exact stationary solution of system (31) is

$$\rho^*(\boldsymbol{q},\boldsymbol{p})=\frac{\rho(\boldsymbol{q},\boldsymbol{p})}{D(\boldsymbol{q})} \tag{32}$$

where $\rho(\boldsymbol{q},\boldsymbol{p})$ is the exact stationary solution of system (31) with $D(\boldsymbol{Q})=1$. Besides, some exact steady state solutions of integrable Hamiltonian system subject to both harmonic and Gaussian white noise excitations have been obtained [4,18].

It is noted that solution (11) has the property of energy equipartition among various degrees of freedom of the system and only the total energy of the system is controlled by the damping forces and stochastic excitations. Solutions (15), (19), (23) and (27), which are consistent with the solution of linear systems under external excitations of Gaussian white noises, on the other hand, have the property of energy non-equipatition since both the total energy of the system and its partition among various degrees of freedom can be adjusted by the magnitudes and distributions of the damping forces and stochastic excitations. All the exact stationary solutions of nonlinear stochastic systems obtained from 1933 up to early 1990s [19-24] are special cases of solution (11). The only exception is the solution obtained by Cai and Lin[25], which is a special case of solution (15). Thus, obtaining solutions (15), (19), (23) and (27) broke the limitations of energy-equipartition solution.

4. Equivalent Nonlinear System Method

The conditions for the exact stationary solutions to exist, such as equations (13), (16), (20),