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INTRODUCTION to MECHANICS and SYMMETRY

力学和对称性导论

Jerrold E. Marsden ■ Tudor S. Ratiu

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Introduction to Mechanics and Symmetry

A Basic Exposition of Classical Mechanical Systems

With 43 Illustrations

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To Barbara and Lilian for their love and support

Series Preface

Mathematics is playing an ever more important role in the physical and biological sciences, provoking a blurring of boundaries between scientific disciplines and a resurgence of interest in the modern as well as the classical techniques of applied mathematics. This renewal of interest, both in research and teaching, has led to the establishment of the series: *Texts in Applied Mathematics (TAM)*.

The development of new courses is a natural consequence of a high level of excitement on the research frontier as newer techniques, such as numerical and symbolic computer systems, dynamical systems, and chaos, mix with and reinforce the traditional methods of applied mathematics. Thus, the purpose of this textbook series is to meet the current and future needs of these advances and encourage the teaching of new courses.

TAM will publish textbooks suitable for use in advanced undergraduate and beginning graduate courses, and will complement the *Applied Mathematical Sciences (AMS)* series, which will focus on advanced textbooks and research level monographs.

Preface

Symmetry and mechanics have been close partners since the time of the founding masters, namely, Newton, Euler, Lagrange, Laplace, Poisson, Jacobi, and Hamilton, and its subsequent developers, including Noether, Riemann, Routh, Kelvin, Poincaré, and Cartan. To this day, symmetry has continued to play a strong role, especially with the modern work of Arnold, Guillemin, Kirillov, Kostant, Moser, Smale, Souriau, Sternberg, and many others. This book is about these developments, with an emphasis on concrete applications that we hope will make it accessible to a wide variety of readers, especially senior undergraduate and graduate students in mathematics, physics, and engineering.

The geometric point of view in mechanics combined with solid analysis has been a phenomenal success in linking various diverse areas, both within and across standard disciplinary lines. It has provided both insight into fundamental issues in mechanics (such as Hamiltonian structures in continuum mechanics, fluid mechanics, and plasma physics) and provided useful tools in specific models such as new stability and bifurcation criteria using the energy-Casimir and energy-momentum methods, new numerical codes based on geometrically exact update procedures, and new reorientation techniques in control theory and robotics.

The role of symmetry in mechanical problems, which was already widely used by the founders of the subject, has been developed considerably in recent times to gain further understanding into such diverse phenomena as reduction, stability, and bifurcation relative to prescribed symmetries (symmetry breaking), methods of finding explicit solutions for integrable systems, and a deeper penetration into special systems, such as the Kowalewski top. We hope this book will provide a reasonable avenue to, and foundation for, these exciting developments.

Because of the extensive and complex set of possible directions in which one can develop the theory, we have provided a fairly lengthy introduction. It is intended to be read lightly at the beginning and then consulted from time to time as the text itself is read. This volume contains much of the basic theory of mechanics and should prove to be a useful foundation for further, as well as more specialized topics. In particular, due to space limitations we warn the reader that many important topics in mechanics are not treated in this volume. We are preparing a second volume on general reduction theory and its applications. With luck and a little support, it will be available in the near future.

A solution manual is available that contains complete solutions to many of the exercises and other supplementary comments. To obtain one, send a mailing label along with \$15 to cover printing and postage to J. Marsden, Department of Mathematics, University of California, Berkeley, CA 94720.

We thank Alan Weinstein, Rudolf Schmid, and Rich Spencer for helping with an early set of notes that helped us on our way. Our many colleagues, students, and readers, especially Henry Abarbanel, Vladimir Arnold, Larry Bates, Michael Berry, Tony Bloch, Marty Golubitsky, Mark Gotay, George Haller, Aaron Hershman, Darryl Holm, Phil Holmes, Sameer Jalnapurkar, Edgar Knobloch, P.S. Krishnaprasad, Debra Lewis, Robert Littlejohn, Richard Montgomery, Phil Morrison, Richard Murray, Oliver O'Reilly, George Patrick, Octavian Popp, Matthias Reinsch, Shankar Sastry, Juan Simo, Hans Troger, and Steve Wiggins have our deepest gratitude for their encouragement and suggestions. We also collectively thank all our students and colleagues who have used these notes and have provided valuable advice. We are also indebted to Carol Cook, Anne Kao, Nawoyuki Gregory Kubota, Sue Knapp, Barbara Marsden, Marnie McElhiney, June Meyermann, Teresa Wild, and Ester Zack for their dedicated and patient work on the typesetting and artwork for this book. We want to single out with special thanks, Nawoyuki Gregory Kubota for his special effort with the typesetting and the figures (including the cover illustration and his skillful use of Mathematica). We also thank the staff at Springer-Verlag, especially Laura Carlson, Ken Dreyhaupt, Rüdiger Gebauer, and Karen Kosztolnyik for their skillful editorial work and production of the book.

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About the Authors

Jerrold E. Marsden is professor of mathematics and EECS at the University of California at Berkeley. He has done extensive research in the area of geometric mechanics, with applications to rigid body systems, fluid mechanics, elasticity theory, and plasma theory, as well as to general field theory, including relativistic fields. He also works in dynamical systems and control theory, especially how it relates to mechanical systems and systems with symmetry. He was one of the original founders in the early 1970s of reduction theory for mechanical systems with symmetry, which remains an active and much-studied area of research today. He was the recipient of the prestigious Norbert Wiener prize of the American Mathematical Society and the Society for Industrial and Applied Mathematics in 1990. He has been a Carnegie Fellow at Heriot-Watt University (1977), a Killam Fellow at the University of Calgary (1979), a Miller Fellow at the University of California, Berkeley (1981–1982), a Humboldt Fellow in Germany (1991), and a Fairchild Fellow at Caltech (1992). He has served in several administrative capacities, such as director of the Research Group in Nonlinear Systems and Dynamics at Berkeley (1984–1986), the National Science Foundation Advisory Panel for Mathematics, the Advisory Committee of the Mathematical Sciences Institute at Cornell, and as director of the Fields Institute (1990–1994). He has served as an editor for Springer-Verlag's Applied Mathematical Sciences series since 1982 and is an editor for Springer-Verlag's *Journal of Nonlinear Science*.

Tudor S. Ratiu is professor of mathematics at the University of California at Santa Cruz. He has previously taught at the University of Michigan, Ann Arbor, as a T.H. Hildebrandt Research Assistant Professor (1980–1983) and at the University of Arizona, Tucson (1983–1987). His research interests center on geometric mechanics, symplectic geometry, global analysis, and infinite dimensional Lie theory, together with their applications to integrable systems, nonlinear dynamics, continuum mechanics, plasma physics, and bifurcation theory. He has been a National Science Foundation Postdoctoral Fellow (1983–1986), an A.P. Sloan Foundation Fellow (1984–1987), and a Miller Research Professor at the University of California at Berkeley (1994). He was a member of the Mathematical Sciences Research Institute in Berkeley (1983–1984, 1989–1990, 1994), the Center of Nonlinear Science at the Los Alamos National Laboratory (1983), the Max Planck Institute of Mathematics in Bonn, Germany (1984, 1990), the Mathematical Sciences Institute at Cornell University (1990), the University of Montpellier, France (1990), the Fields Institute (1993), the Erwin Schrödinger Institute of Mathematical Physics in Vienna, Austria (1993, 1994), and the Institut des Hautes Etudes Scientifiques in Bures-sur-Yvette, France (1994–1995). Since his arrival at UC Santa Cruz in 1987, he has been on the executive committee of the Nonlinear Sciences Organized Research Unit. He is currently on the editorial board of the *Annals of Global Analysis* and the *Annals of the University of Timisoara*.

Contents

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Introduction and Overview

1.1 Lagrangian and Hamiltonian Formalisms

Classical mechanics deals with the dynamics of particles, rigid bodies, continuous media (fluid, plasma, and solid mechanics), and other fields (such as electromagnetism, gravity, etc.). This theory also plays a crucial role in quantum mechanics, in control theory and other areas of physics, engineering and even chemistry and biology. Clearly classical mechanics is a large subject that plays a fundamental role in science. Throughout history, mechanics has also played a key role in the development of mathematics. Starting with the creation of calculus stimulated by Newton's mechanics, it continues today with exciting developments in group representations, geometry, and topology; these mathematical developments in turn are being applied to interesting problems in physics and engineering.

Symmetry has always played an important role in mechanics, from fundamental formulations of basic principles to concrete applications, such as stability criteria for rotating structures. The theme of this book is to emphasize the role of symmetry in various aspects of mechanics.

Warning This introduction treats a collection of topics fairly rapidly. The student should not expect to understand everything perfectly at this stage. *We will return to many of the topics in subsequent chapters.*

Mechanics has two main branches, ***Lagrangian mechanics*** and ***Hamiltonian mechanics***. In one sense, Lagrangian mechanics is more fundamental since it is based on variational principles and it is what generalizes most directly to the general relativistic context. In another sense, Hamiltonian mechanics is more fundamental, since it is based directly on the energy concept and it is what is more closely tied to quantum mechanics. Fortunately, in many cases these branches are equivalent as we shall see in detail in Chapter 7. Needless to say, the merger of quantum mechanics and general relativity remains one of the main problems of mechanics.

The Lagrangian formulation of mechanics can be based on the observation that there are variational principles behind the fundamental laws of force balance as given by Newton's law in $\mathbf{F} = m\mathbf{a}$. One chooses a configuration space Q with coordinates $q^i, i = 1, \ldots, n$, that describe the ***configuration*** of the system under study. Then one introduces the ***Lagrangian*** $L(q^i, \dot{q}^i, t)$, which is shorthand notation for $L(q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)$. Usually, L is the kinetic *minus* the potential energy of the system and one takes $\dot{q}^i = dq^i/dt$ regarded as the velocity. The ***variational principle of Hamilton*** states

$$\delta \int_a^b L(q^i, \dot{q}^i, t)\, dt = 0. \tag{1.1.1}$$

In (1.1.1), we choose curves $q^i(t)$ joining two fixed points in Q over a fixed time interval $[a, b]$, and calculate the integral regarded as a function of this curve. Then (1.1.1) states that this function has a critical point. If we let δq^i be a variation of the curve (and proceed somewhat formally at first), then by the chain rule, (1.1.1) is equivalent to

$$\sum_{i=1}^{n} \int_a^b \left(\frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \delta \dot{q}^i \right) dt = 0 \tag{1.1.2}$$

for all variations δq^i.

Using $\delta \dot{q}^i = \frac{d}{dt} \delta q^i$ (which is essentially the equality of mixed partials), integrating the second term by parts, and using the boundary conditions $\delta q^i = 0$ at $t = a$ and b, (1.1.2) becomes

$$\sum_{i=1}^{n} \int_a^b \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) \right) \delta q^i\, dt = 0. \tag{1.1.3}$$

Since δq^i is arbitrary (apart from being zero at the endpoints), (1.1.2) is equivalent to the ***Euler-Lagrange equations***

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \ldots, n. \tag{1.1.4}$$

(This topic will be discussed at greater length in §7.3 and §8.1). For the case of kinetic minus potential energy for a system of particles, where L has the form

$$L(q^i, \dot{q}^i, t) = \frac{1}{2} \sum_{i=1}^{n} m_i \|\dot{q}^i\|^2 - V(q^i), \tag{1.1.5}$$

(1.1.4) reduces to

$$\frac{d}{dt}(m_i \dot{q}^i) = -\frac{\partial V}{\partial q^i}, \tag{1.1.6}$$

which is $\mathbf{F} = m\mathbf{a}$ for the motion of a particle in the potential field V.

Already at this stage, interesting links with geometry emerge. If $g_{ij}(q)$ is a given metric tensor (for now, just think of this as a q-dependent positive-definite symmetric $n \times n$ matrix) and we consider the kinetic energy Lagrangian

$$L(q^i, \dot{q}^i) = \frac{1}{2} \sum_{i,j=1}^{n} g_{ij}(q) \dot{q}^i \dot{q}^j, \tag{1.1.7}$$

then *the Euler-Lagrange equations are equivalent to the equations of geodesic motion*, as can be directly verified (see §7.5 for details). Conservation laws that are a result of symmetry in a mechanical context can then be applied to yield interesting geometric facts. For instance, we will see that theorems about geodesics on surfaces of revolution can be readily proved this way.

The Lagrangian formalism can be extended to the infinite dimensional case. Here the q^i are replaced by *fields* $\varphi^1, \ldots, \varphi^m$ which are, for example, functions of spatial points x^i and time. Then L is a function of $\varphi^1, \ldots, \varphi^m, \dot{\varphi}^1, \ldots, \dot{\varphi}^m$ and the spatial derivatives of the fields. We shall deal with various examples of this later, but we emphasize that properly interpreted, the variational principle and the Euler-Lagrange equations remain intact. One simply replaces the partial derivatives in the Euler-Lagrange equations by *functional derivatives* defined below.

To pass to the Hamiltonian formalism, introduce the ***conjugate momenta***

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \qquad i = 1, \ldots, n, \tag{1.1.8}$$

make the change of variables $(q^i, \dot{q}^i) \mapsto (q^i, p_i)$, and introduce the Hamiltonian

$$H(q^i, p_i, t) = \sum_{j=1}^{n} p_j \dot{q}^j - L(q^i, \dot{q}^i, t). \tag{1.1.9}$$

Remembering the change of variables, we make these computations:

$$\frac{\partial H}{\partial p_i} = \dot{q}^i + \sum_{j=1}^{n} \left(p_j \frac{\partial \dot{q}^j}{\partial p_i} - \frac{\partial L}{\partial \dot{q}^j} \frac{\partial \dot{q}^j}{\partial p_i} \right) = \dot{q}^i \tag{1.1.10}$$

and

$$\frac{\partial H}{\partial q^i} = \sum_{j=1}^{n} p_j \frac{\partial \dot{q}^j}{\partial q^i} - \frac{\partial L}{\partial q^i} - \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}^j} \frac{\partial \dot{q}^j}{\partial q^i} = -\frac{\partial L}{\partial q^i}, \tag{1.1.11}$$

where (1.1.8) has been used twice. Using (1.1.4) and (1.1.8), we see that (1.1.11) is equivalent to

$$\frac{\partial H}{\partial q^i} = -\frac{d}{dt}p_i. \tag{1.1.12}$$

Thus, *the Euler-Lagrange equations are equivalent to* ***Hamilton's equations***

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}, \quad i = 1, \ldots, n. \tag{1.1.13}$$

The analogous Hamiltonian partial differential equations for time dependent *fields* $\varphi^1, \ldots, \varphi^m$ and their conjugate momenta $\pi_1, \ldots, \pi_m$, are

$$\frac{\partial \varphi^a}{\partial t} = \frac{\delta H}{\partial \pi_a}, \qquad \frac{\partial \pi_a}{\partial t} = -\frac{\delta H}{\partial \varphi^a}, \quad a = 1, \ldots, m, \tag{1.1.14}$$

where H is a functional of the fields φ^a and π_a, and the ***variational*** or ***functional derivatives*** are defined through the equation

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\delta H}{\delta \varphi^1} \delta\varphi^1 \, d^n x \quad = \quad & \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[H(\varphi^1 + \varepsilon\delta\varphi^1, \varphi^2, \ldots, \varphi^m, \pi_1, \ldots, \pi_m) \\ & - H(\varphi^1, \varphi^2, \ldots, \varphi^m, \pi_1, \ldots, \pi_m)],\end{aligned} \tag{1.1.15}$$

and similarly for $\delta H/\delta\varphi^2, \ldots, \delta H/\delta\pi_m$. Both equations (1.1.13) and (1.1.14) can be recast in ***Poisson bracket form***

$$\dot{F} = \{F, H\}, \tag{1.1.16}$$

where the brackets in the respective cases are given by

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q^i} \right) \tag{1.1.17}$$

and

$$\{F, G\} = \sum_{a=1}^{m} \int_{\mathbb{R}^n} \left(\frac{\delta F}{\delta \varphi^a} \frac{\delta G}{\delta \pi_a} - \frac{\delta F}{\delta \pi_a} \frac{\delta G}{\delta \varphi^a} \right) d^n x. \tag{1.1.18}$$

There is also a variational principle valid directly on the Hamiltonian side. For the Euler-Lagrange equations, we deal with curves in q-space, whereas for Hamilton's equations we deal with curves in (q, p)-space. The principle is

$$\delta \int_a^b \sum_{i=1}^{n} [p_i \dot{q}^i - H(q^j, p_j)] \, dt = 0 \tag{1.1.19}$$

as is readily verified; one requires $p_i \delta q^i = 0$ at the endpoints.

This formalism is the basis for the analysis of many important systems in particle dynamics and field theory, as described in standard texts such