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PRINCETON LECTURES IN ANALYSIS I

FOURIER ANALYSIS AN INTRODUCTION

傅立叶分析导论

ELIAS M. STEIN & RAMI SHAKARCHI

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I

FOURIER ANALYSIS

AN INTRODUCTION

Elias M. Stein

&

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TO MY GRANDCHILDREN
CAROLYN, ALISON, JASON

E.M.S.

TO MY PARENTS
MOHAMED & MIREILLE
AND MY BROTHER
KARIM

R.S.

Foreword

Beginning in the spring of 2000, a series of four one-semester courses were taught at Princeton University whose purpose was to present, in an integrated manner, the core areas of analysis. The objective was to make plain the organic unity that exists between the various parts of the subject, and to illustrate the wide applicability of ideas of analysis to other fields of mathematics and science. The present series of books is an elaboration of the lectures that were given.

While there are a number of excellent texts dealing with individual parts of what we cover, our exposition aims at a different goal: presenting the various sub-areas of analysis not as separate disciplines, but rather as highly interconnected. It is our view that seeing these relations and their resulting synergies will motivate the reader to attain a better understanding of the subject as a whole. With this outcome in mind, we have concentrated on the main ideas and theorems that have shaped the field (sometimes sacrificing a more systematic approach), and we have been sensitive to the historical order in which the logic of the subject developed.

We have organized our exposition into four volumes, each reflecting the material covered in a semester. Their contents may be broadly summarized as follows:

- I. Fourier series and integrals.
- II. Complex analysis.
- III. Measure theory, Lebesgue integration, and Hilbert spaces.
- IV. A selection of further topics, including functional analysis, distributions, and elements of probability theory.

However, this listing does not by itself give a complete picture of the many interconnections that are presented, nor of the applications to other branches that are highlighted. To give a few examples: the elements of (finite) Fourier series studied in Book I, which lead to Dirichlet characters, and from there to the infinitude of primes in an arithmetic progression; the X -ray and Radon transforms, which arise in a number of

problems in Book I, and reappear in Book III to play an important role in understanding Besicovitch-like sets in two and three dimensions; Fatou's theorem, which guarantees the existence of boundary values of bounded holomorphic functions in the disc, and whose proof relies on ideas developed in each of the first three books; and the theta function, which first occurs in Book I in the solution of the heat equation, and is then used in Book II to find the number of ways an integer can be represented as the sum of two or four squares, and in the analytic continuation of the zeta function.

A few further words about the books and the courses on which they were based. These courses were given at a rather intensive pace, with 48 lecture-hours a semester. The weekly problem sets played an indispensable part, and as a result exercises and problems have a similarly important role in our books. Each chapter has a series of "Exercises" that are tied directly to the text, and while some are easy, others may require more effort. However, the substantial number of hints that are given should enable the reader to attack most exercises. There are also more involved and challenging "Problems"; the ones that are most difficult, or go beyond the scope of the text, are marked with an asterisk.

Despite the substantial connections that exist between the different volumes, enough overlapping material has been provided so that each of the first three books requires only minimal prerequisites: acquaintance with elementary topics in analysis such as limits, series, differentiable functions, and Riemann integration, together with some exposure to linear algebra. This makes these books accessible to students interested in such diverse disciplines as mathematics, physics, engineering, and finance, at both the undergraduate and graduate level.

It is with great pleasure that we express our appreciation to all who have aided in this enterprise. We are particularly grateful to the students who participated in the four courses. Their continuing interest, enthusiasm, and dedication provided the encouragement that made this project possible. We also wish to thank Adrian Banner and Jose Luis Rodrigo for their special help in running the courses, and their efforts to see that the students got the most from each class. In addition, Adrian Banner also made valuable suggestions that are incorporated in the text.

We wish also to record a note of special thanks for the following individuals: Charles Fefferman, who taught the first week, (successfully launching the whole project!); Paul Hagelstein, who in addition to reading part of the manuscript taught several weeks of one of the courses, and has since taken over the teaching of the second round of the series; Daniel Levine who gave valuable help in proof-reading. Last but not least, our thanks go to Gerree Pecht, for her consummate skill in typesetting and for the time and energy she spent in the preparation of all aspects of the lectures, such as transparencies, notes, and the manuscript.

We are also happy to acknowledge our indebtedness for the support we received from the 250th Anniversary Fund of Princeton University, and the National Science Foundation's VIGRE program.

Elias M. Stein
Rami Shakarchi

Princeton, New Jersey
August 2002

Preface to Book I

Any effort to present an overall view of analysis must at its start deal with the following questions: Where does one begin? What are the initial subjects to be treated, and in what order are the relevant concepts and basic techniques to be developed?

Our answers to these questions are guided by our view of the centrality of Fourier analysis, both in the role it has played in the development of the subject, and in the fact that its ideas permeate much of the present-day analysis. For these reasons we have devoted this first volume to an exposition of some basic facts about Fourier series, taken together with a study of elements of Fourier transforms and finite Fourier analysis. Starting this way allows one to see rather easily certain applications to other sciences, together with the link to such topics as partial differential equations and number theory. In later volumes several of these connections will be taken up from a more systematic point of view, and the ties that exist with complex analysis, real analysis, Hilbert space theory, and other areas will be explored further.

In the same spirit, we have been mindful not to overburden the beginning student with some of the difficulties that are inherent in the subject: a proper appreciation of the subtleties and technical complications that arise can come only after one has mastered some of the initial ideas involved. This point of view has led us to the following choice of material in the present volume:

- Fourier series. At this early stage it is not appropriate to introduce measure theory and Lebesgue integration. For this reason our treatment of Fourier series in the first four chapters is carried out in the context of Riemann integrable functions. Even with this restriction, a substantial part of the theory can be developed, detailing convergence and summability; also, a variety of connections with other problems in mathematics can be illustrated.
- Fourier transform. For the same reasons, instead of undertaking the theory in a general setting, we confine ourselves in Chapters 5 and 6 largely to the framework of test functions. Despite these limitations, we can learn a number of basic and interesting facts about Fourier analysis in $\mathbb{R}^d$ and its relation to other areas, including the wave equation and the Radon transform.

- Finite Fourier analysis. This is an introductory subject *par excellence*, because limits and integrals are not explicitly present. Nevertheless, the subject has several striking applications, including the proof of the infinitude of primes in arithmetic progression.

Taking into account the introductory nature of this first volume, we have kept the prerequisites to a minimum. Although we suppose some acquaintance with the notion of the Riemann integral, we provide an appendix that contains most of the results about integration needed in the text.

We hope that this approach will facilitate the goal that we have set for ourselves: to inspire the interested reader to learn more about this fascinating subject, and to discover how Fourier analysis affects decisively other parts of mathematics and science.

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1 The Genesis of Fourier Analysis

Regarding the researches of d'Alembert and Euler could one not add that if they knew this expansion, they made but a very imperfect use of it. They were both persuaded that an arbitrary and discontinuous function could never be resolved in series of this kind, and it does not even seem that anyone had developed a constant in cosines of multiple arcs, the first problem which I had to solve in the theory of heat.

J. Fourier, 1808-9

In the beginning, it was the problem of the vibrating string, and the later investigation of heat flow, that led to the development of Fourier analysis. The laws governing these distinct physical phenomena were expressed by two different partial differential equations, the wave and heat equations, and these were solved in terms of Fourier series.

Here we want to start by describing in some detail the development of these ideas. We will do this initially in the context of the problem of the vibrating string, and we will proceed in three steps. First, we describe several physical (empirical) concepts which motivate corresponding mathematical ideas of importance for our study. These are: the role of the functions $\cos t$, $\sin t$, and e^{it} suggested by simple harmonic motion; the use of separation of variables, derived from the phenomenon of standing waves; and the related concept of linearity, connected to the superposition of tones. Next, we derive the partial differential equation which governs the motion of the vibrating string. Finally, we will use what we learned about the physical nature of the problem (expressed mathematically) to solve the equation. In the last section, we use the same approach to study the problem of heat diffusion.

Given the introductory nature of this chapter and the subject matter covered, our presentation cannot be based on purely mathematical reasoning. Rather, it proceeds by plausibility arguments and aims to provide the motivation for the further rigorous analysis in the succeeding chapters. The impatient reader who wishes to begin immediately with the theorems of the subject may prefer to pass directly to the next chapter.

1 The vibrating string

The problem consists of the study of the motion of a string fixed at its end points and allowed to vibrate freely. We have in mind physical systems such as the strings of a musical instrument. As we mentioned above, we begin with a brief description of several observable physical phenomena on which our study is based. These are:

- simple harmonic motion,
- standing and traveling waves,
- harmonics and superposition of tones.

Understanding the empirical facts behind these phenomena will motivate our mathematical approach to vibrating strings.

Simple harmonic motion

Simple harmonic motion describes the behavior of the most basic oscillatory system (called the simple harmonic oscillator), and is therefore a natural place to start the study of vibrations. Consider a mass $\{m\}$ attached to a horizontal spring, which itself is attached to a fixed wall, and assume that the system lies on a frictionless surface.

Choose an axis whose origin coincides with the center of the mass when it is at rest (that is, the spring is neither stretched nor compressed), as shown in Figure 1. When the mass is displaced from its initial equilibrium

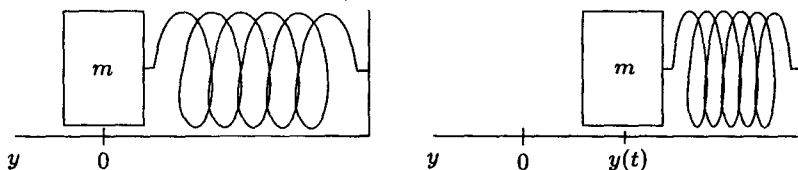


Figure 1. Simple harmonic oscillator

position and then released, it will undergo **simple harmonic motion**. This motion can be described mathematically once we have found the differential equation that governs the movement of the mass.

Let $y(t)$ denote the displacement of the mass at time t . We assume that the spring is ideal, in the sense that it satisfies Hooke's law: the restoring force F exerted by the spring on the mass is given by $F = -ky(t)$. Here

$k > 0$ is a given physical quantity called the spring constant. Applying Newton's law (force = mass $\times$ acceleration), we obtain

$$-ky(t) = my''(t),$$

where we use the notation y'' to denote the second derivative of y with respect to t . With $c = \sqrt{k/m}$, this second order ordinary differential equation becomes

$$(1) \quad y''(t) + c^2 y(t) = 0.$$

The general solution of equation (1) is given by

$$y(t) = a \cos ct + b \sin ct,$$

where a and b are constants. Clearly, all functions of this form solve equation (1), and Exercise 6 outlines a proof that these are the only (twice differentiable) solutions of that differential equation.

In the above expression for $y(t)$, the quantity c is given, but a and b can be any real numbers. In order to determine the particular solution of the equation, we must impose two initial conditions in view of the two unknown constants a and b . For example, if we are given $y(0)$ and $y'(0)$, the initial position and velocity of the mass, then the solution of the physical problem is unique and given by

$$y(t) = y(0) \cos ct + \frac{y'(0)}{c} \sin ct.$$

One can easily verify that there exist constants $A > 0$ and $\varphi \in \mathbb{R}$ such that

$$a \cos ct + b \sin ct = A \cos(ct - \varphi).$$

Because of the physical interpretation given above, one calls $A = \sqrt{a^2 + b^2}$ the "amplitude" of the motion, c its "natural frequency," φ its "phase" (uniquely determined up to an integer multiple of 2π), and $2\pi/c$ the "period" of the motion.

The typical graph of the function $A \cos(ct - \varphi)$, illustrated in Figure 2, exhibits a wavelike pattern that is obtained from translating and stretching (or shrinking) the usual graph of $\cos t$.

We make two observations regarding our examination of simple harmonic motion. The first is that the mathematical description of the most elementary oscillatory system, namely simple harmonic motion, involves