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Positive Solutions

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Johnny Henderson dedicates this book to his siblings, Monty, Madonna, Jana, and Chrissie, and to the memory of his parents, Ernest and Madora. Rodica Luca dedicates this book to her husband, Mihai Tudorache, and her son, Alexandru-Gabriel Tudorache, and to the memory of her parents, Viorica and Constantin Luca.

Preface

In recent decades, nonlocal boundary value problems for ordinary differential equations, difference equations, or fractional differential equations have become a rapidly growing area of research. The study of these types of problems is driven not only by a theoretical interest, but also by the fact that several phenomena in engineering, physics, and the life sciences can be modeled in this way. Boundary value problems with positive solutions describe many phenomena in the applied sciences such as the nonlinear diffusion generated by nonlinear sources, thermal ignition of gases, and concentration in chemical or biological problems. Various problems arising in heat conduction, underground water flow, thermoelasticity, and plasma physics can be reduced to nonlinear differential problems with integral boundary conditions. Fractional differential equations describe many phenomena in several fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology (such as blood flow phenomena), economics, control theory, signal and image processing, aerodynamics, viscoelasticity, and electromagnetics.

Hundreds of researchers are working on boundary value problems for differential equations, difference equations, and fractional equations, and at the heart of the community are researchers whose interest is in positive solutions. The authors of this monograph occupy a niche in the center of that group. The monograph contains many of their results related to these topics obtained in recent years.

In Chapter 1, questions are addressed on the existence, multiplicity, and nonexistence of positive solutions for some classes of systems of nonlinear second-order ordinary differential equations with parameters or without parameters, subject to Riemann–Stieltjes boundary conditions, and for which the nonlinearities are nonsingular or singular functions. Chapter 2 is devoted to the existence, multiplicity, and nonexistence of positive solutions for some classes of systems of nonlinear higher-order ordinary differential equations with parameters or without parameters, subject to multipoint boundary conditions, and for which the nonlinearities are nonsingular or singular functions. A system of higher-order differential equations with sign-changing nonlinearities and Riemann–Stieltjes integral boundary conditions is also investigated. Chapter 3 deals with the existence, multiplicity, and nonexistence of positive solutions for some classes of systems of nonlinear second-order difference equations, also with or without parameters, subject to multipoint boundary conditions.

Chapter 4 is concerned with the existence, multiplicity, and nonexistence of positive solutions for some classes of systems of nonlinear Riemann–Liouville fractional differential equations with parameters or without parameters, subject to uncoupled Riemann–Stieltjes integral boundary conditions, and for which the nonlinearities are nonsingular or singular functions. A system of fractional equations with sign-changing nonlinearities and integral boundary conditions is also investigated. Chapter 5 is focused on the existence, multiplicity, and nonexistence of positive solutions for some classes of systems of nonlinear Riemann–Liouville fractional differential equations with parameters or without parameters, subject to coupled Riemann–Stieltjes integral boundary conditions, and for which the nonlinearities are nonsingular or singular functions. A system of fractional equations with sign-changing nonsingular or singular nonlinearities and integral boundary conditions is also investigated. In each chapter, various examples are presented which support the main results.

Central to the results of each chapter are applications of the Guo–Krasnosel'skii fixed point theorem for nonexpansive and noncontractive operators on a cone (Theorem 1.1.1). Unique to applications of the fixed point theorem is the novel representation of the Green's functions, which ultimately provide almost a checklist for determining conditions for which positive solutions exist relative to given nonlinearities. In the proof of many of the main results, applications are also made of the Schauder fixed-point theorem (Theorem 1.6.1), the nonlinear alternative of Leray–Schauder type (Theorem 2.5.1), and some theorems from the fixed point index theory (Theorems 1.3.1–1.3.3).

There have been other books in the past on positive solutions for boundary value problems, but in spite of the area receiving much attention, there have been no new books recently. This monograph provides a springboard for other researchers to emulate the authors' methods. The audience for this book includes the family of mathematical and scientific researchers in boundary value problems for which positive solutions are important, and in addition, the monograph can serve as a great source for topics to be studied in graduate seminars.

Johnny Henderson
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About the authors

Johnny Henderson is a distinguished professor of Mathematics at the Baylor University, Waco, Texas, USA. He has also held faculty positions at the Auburn University and the Missouri University of Science and Technology. His published research is primarily in the areas of boundary value problems for ordinary differential equations, finite difference equations, functional differential equations, and dynamic equations on time scales. He is an Inaugural Fellow of the American Mathematical Society.

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Contents

Systems of second-order ordinary differential equations with integral boundary conditions

1

1.1 Existence of positive solutions for systems with parameters

Boundary value problems with positive solutions describe many phenomena in the applied sciences, such as the nonlinear diffusion generated by nonlinear sources, thermal ignition of gases, and concentration in chemical or biological problems (see Boucherif and Henderson, 2006; Cac et al., 1997; de Figueiredo et al., 1982; Guo and Lakshmikantham, 1988a,b; Joseph and Sparrow, 1970; Keller and Cohen, 1967). Integral boundary conditions arise in thermal conduction, semiconductor, and hydrodynamic problems (e.g., Cannon, 1964; Chegis, 1984; Ionkin, 1977; Samarskii, 1980). In recent decades, many authors have investigated scalar problems with integral boundary conditions (e.g., Ahmad et al., 2008; Boucherif, 2009; Jankowski, 2013; Jia and Wang, 2012; Karakostas and Tsamatos, 2002; Ma and An, 2009; Webb and Infante, 2008; Yang, 2006). We also mention references (Cui and Sun, 2012; Goodrich, 2012; Hao et al., 2012; Infante et al., 2012; Infante and Pietramala, 2009a,b; Kang and Wei, 2009; Lan, 2011; Song and Gao, 2011; Yang, 2005; Yang and O'Regan, 2005; Yang and Zhang, 2012), where the authors studied the existence of positive solutions for some systems of differential equations with integral boundary conditions.

1.1.1 Presentation of the problem

In this section, we consider the system of nonlinear second-order ordinary differential equations

$$\begin{cases} (a(t)u'(t))' - b(t)u(t) + \lambda p(t)f(t,u(t),v(t)) = 0, & 0 < t < 1, \\ (c(t)v'(t))' - d(t)v(t) + \mu q(t)g(t,u(t),v(t)) = 0, & 0 < t < 1, \end{cases} \tag{S}$$

with the integral boundary conditions

$$\begin{cases} \alpha u(0) - \beta a(0)u'(0) = \int_0^1 u(s)\,\mathrm{d}H_1(s), & \gamma u(1) + \delta a(1)u'(1) = \int_0^1 u(s)\,\mathrm{d}H_2(s), \\ \tilde{\alpha} v(0) - \tilde{\beta} c(0)v'(0) = \int_0^1 v(s)\,\mathrm{d}K_1(s), & \tilde{\gamma} v(1) + \tilde{\delta} c(1)v'(1) = \int_0^1 v(s)\,\mathrm{d}K_2(s), \end{cases} \tag{BC}$$

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where the above integrals are Riemann–Stieltjes integrals. The boundary conditions above include multipoint and integral boundary conditions and the sum of these in a single framework.

We give sufficient conditions on f and g and on the parameters λ and μ such that positive solutions of problem (S)–(BC) exist. By a positive solution of problem (S)–(BC) we mean a pair of functions $(u, v) \in C^2([0,1]) \times C^2([0,1])$ satisfying (S) and (BC) with $u(t) \geq 0$, $v(t) \geq 0$ for all $t \in [0,1]$ and $(u, v) \neq (0,0)$. The case in which the functions H_1, H_2, K_1, and K_2 are step functions—that is, the boundary conditions (BC) become multipoint boundary conditions

$$\begin{cases} \alpha u(0) - \beta a(0)u'(0) = \sum_{i=1}^{m} a_i u(\xi_i), & \gamma u(1) + \delta a(1)u'(1) = \sum_{i=1}^{n} b_i u(\eta_i), \\ \tilde{\alpha} v(0) - \tilde{\beta} c(0)v'(0) = \sum_{i=1}^{r} c_i v(\zeta_i), & \tilde{\gamma} v(1) + \tilde{\delta} c(1)v'(1) = \sum_{i=1}^{l} d_i v(\rho_i), \end{cases} \tag{BC$_1$}$$

where $m, n, r, l \in \mathbb{N}$, $(\mathbb{N} = \{1, 2, \ldots\})$—was studied in Henderson and Luca (2013g). System (S) with $a(t) = 1$, $c(t) = 1$, $b(t) = 0$, and $d(t) = 0$ for all $t \in [0,1]$, $f(t,u,v) = \tilde{f}(u,v)$, and $g(t,u,v) = \tilde{g}(u,v)$ (denoted by (S$_1$)) and (BC$_1$) was investigated in Henderson and Luca (2014e). Some particular cases of the problem from Henderson and Luca (2014e) were studied in Henderson and Luca (2012e) (where in (BC$_1$), $a_i = 0$ for all $i = 1, \ldots, m$, $c_i = 0$ for all $i = 1, \ldots, r$, $\gamma = \tilde{\gamma} = 1$, and $\delta = \tilde{\delta} = 0$—denoted by (BC$_2$)), in Luca (2011) (where in (S$_1$), $\tilde{f}(u,v) = \tilde{f}(v)$ and $\tilde{g}(u,v) = \tilde{g}(u)$—denoted by (S$_2$)—and in (BC$_2$) we have $n = l$, $b_i = d_i$, and $\eta_i = \rho_i$ for all $i = 1, \ldots, n$, $\alpha = \tilde{\alpha}$, and $\beta = \tilde{\beta}$—denoted by (BC$_3$)), in Henderson et al. (2008b) (problem (S$_2$)–(BC$_3$) with $\alpha = \tilde{\alpha} = 1$ and $\beta = \tilde{\beta} = 0$), and in Henderson and Ntouyas (2008b) and Henderson et al. (2008a) (system (S$_2$) with the boundary conditions $u(0) = 0$, $u(1) = \alpha u(\eta)$, $v(0) = 0$, $v(1) = \alpha v(\eta)$, $\eta \in (0,1)$, and $0 < \alpha < 1/\eta$, or $u(0) = \beta u(\eta)$, $u(1) = \alpha u(\eta)$, $v(0) = \beta v(\eta)$, and $v(1) = \alpha v(\eta)$). In Henderson and Ntouyas (2008a), the authors investigated system (S$_2$) with the boundary conditions $\alpha u(0) - \beta u'(0) = 0$, $\gamma u(1) + \delta u'(1) = 0$, $\alpha v(0) - \beta v'(0) = 0$, and $\gamma v(1) + \delta v'(1) = 0$, where $\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha + \beta + \gamma + \delta > 0$.

In the proof of our main results, we shall use the Guo–Krasnosel'skii fixed point theorem (see Guo and Lakshmikantham, 1988a), which we present now:

Theorem 1.1.1. *Let X be a Banach space and let $C \subset X$ be a cone in X. Assume Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let $\mathcal{A} : C \cap (\overline{\Omega_2} \setminus \Omega_1) \to C$ be a completely continuous operator (continuous, and compact—that is, it maps bounded sets into relatively compact sets) such that either*

(1) $\|\mathcal{A}u\| \leq \|u\|$, $u \in C \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \geq \|u\|$, $u \in C \cap \partial\Omega_2$, *or*
(2) $\|\mathcal{A}u\| \geq \|u\|$, $u \in C \cap \partial\Omega_1$, *and* $\|\mathcal{A}u\| \leq \|u\|$, $u \in C \cap \partial\Omega_2$.

Then, $\mathcal{A}$ has a fixed point in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.