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Discrete Integrable Geometry and Physics

Edited by

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Preface

Many of the fundamental ideas and concrete results of the modern theory of completely integrable systems go back far into the past, and many of them can be identified within the setting of classical differential geometry. In recent years, there has been increasing interest in discrete integrable models — both classical and quantum mechanical — with discretized space and time variables. In particular, a connection between discrete integrable systems and the geometry of polyhedral surfaces and discrete curves has been discovered. It is surprisingly close to the above-mentioned relation between the geometry of curves and surfaces and soliton systems. Hence discrete integrable models and their corresponding geometries may be considered more fundamental than their smooth counterparts, which one may obtain by suitable continuum limits.

Discrete integrable models are related not only to concepts of discrete geometry but also to interesting structures in discrete quantum field theory. In particular it appears that in some cases quantum integrals or charges in the discrete sine-Gordon theory can be identified with magnetic Schrödinger operators with a complex spectral structure, which can be determined by applying Bethe-Ansatz methods.

The present book originated from the results achieved in the special research project SFB 288 "Differential Geometry and Quantum Physics" and results presented at the conference "Condensed Matter Physics and Discrete Geometry" organised by the editors and held at the Erwin Schrödinger Institute in Vienna in February 1996. The intention of the conference at the Erwin Schrödinger Institute, of the SFB project and of this book is the same: to combine the efforts of mathematicians and physicists studying the geometry and physics of discrete integrable systems. Indeed, in many cases they investigate the same models using different methods. The exchange of ideas, methods and models turned out to be fruitful.

The book presents the results of this communication and cooperation. It consists of a sequence of invited expositions by experts in the field, which we hope forms a coherent account of the theory. The purpose is two-fold: first, to explain the ideas and methods starting from an elementary level, and secondly, to bring the reader close to the current state of research in this area.

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R.S.
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Introduction

Alexander I. Bobenko and Ruedi Seiler

Discrete integrable geometry is a newly emerging field of mathematics with strong ties to physics and considerable potential for computer graphics. Our book intends to introduce this field, to present its current status, and to lead to the questions presently under investigation. There is, however, no complete theory about this subject yet. For that reason much of the book is devoted to examples.

Without going into detail about the long and interesting history of integrable systems (the origin of discrete integrable geometry) we would like to mention that in the modern theory of completely integrable systems, geometric concepts and concepts of algebraic geometry play a key role and that many of the most important ideas go back far into the past, and many of them can be found in the classic books by Darboux and Bianchi.

Historically, the relationship between geometry and integrable partial differential equations emerged more than a century ago from the discovery that the sine-Gordon equation

$$\phi_{xy} + \sin\phi = 0 \tag{0.1}$$

describes surfaces with constant negative Gaussian curvature.

Today this equation is the paradigm of non-linear integrable partial differential equations. Equations of this type often appear as compatibility conditions

$$U_y(\lambda) - V_x(\lambda) + [U(\lambda), V(\lambda)] = 0 \tag{0.2}$$

of a linear system of differential equations

$$\Psi_x = U(\lambda)\Psi, \qquad \Psi_y = V(\lambda)\Psi. \tag{0.3}$$

In differential geometry, Ψ has a natural interpretation as a moving frame attached to the surface in Euclidean space. Equation (0.2) is called the Lax representation of equation (0.1). The linear equations (0.3) relate the frames at neighboring points. The spectral parameter λ parameterizes a family of surfaces of constant negative curvature which have the same characteristic features (for details see the contribution of Bobenko and Pinkall; Chapter 1).

After the discovery that the sine-Gordon equation analytically describes constant negative Gaussian curvature surfaces, similar equations were found for other classes of surfaces. From the very beginning it was noticed that these equations have remarkable solutions corresponding to certain special surfaces.

Later it became clear that these equations form some of the most fascinating examples of infinite-dimensional integrable systems. The special solutions mentioned above became known as soliton solutions.

As a side remark we mention that, surprisingly, long before the theory of solitons, geometers sometimes used the modern terminology and called some equations "completely integrable" [1] (which does not, however, coincide with the precise definition that is used today in the theory of Hamiltonian systems). At that time, the Hamiltonian interpretation and, more generally, any physical application of the integrable models studied in geometry, were unknown. Although the Hamiltonian interpretation is not very essential from the point of view of the classical differential geometry of curves and surfaces, in its modern setting of the R-matrix approach it is closely related to the geometrical description in the form of the Lax representation.

Discretization appeared in the geometry of surfaces in the 19th century, albeit in a sense which is slightly different from the one used today. Its level of abstraction proves, however, the impressive depth of understanding of the geometers at that time. They considered the existence of special transformations as one of the crucial properties in order to single out geometries of special interest. One famous example of such a transformation is the Bäcklund transformation for surfaces with constant negative Gaussian curvature. For many geometries—nowadays called integrable—it was shown that Bäcklund transformations are permutable, i.e. there is a natural homomorphism from the square lattice (the free abelian group with two generators) to the Bäcklund transformations. In particular it was Bianchi who stressed the importance of permutability. This group property of the Bäcklund transformation has been completely characterized only recently in terms of loop groups in the theory of solitons. In these terms, Bäcklund transformations act as a multiplication of the frame Ψ by an element of the loop group. They are sometimes called a dressing transformation. It is remarkable that one of the last books of this "classical" period of differential geometry has the title *Transformations of Surfaces*, written by Eisenhart.

A different and entirely independent approach to discrete integrable systems—classical and quantum—originates from the analysis of exactly solvable models in statistical mechanics. The solution of the first model of this class, the Ising model, could be related to its integrability only in retrospect. Thus it was not until the seminal work of Lieb on the ice model, sometimes called the 6-vertex model, that integrability became the key property for solving models of statistical mechanics. Using the results by Yang and Yang on the ground state energy of the Heisenberg chain, Lieb computed the residual entropy at zero temperature. The basic method for the analysis of the spectrum of the Heisenberg chain is the Bethe Ansatz. Its algebraic version, developed mainly in Leningrad, can be

[1] For example, Tzitzéica in *Sur une nouvelle classe de surfaces*, C.R. Acad. Sci. Paris, **150** (1910) 955–956, 1227–1229 discusses "système complètement intégrable" in affine differential geometry.

used to compute spectral properties of quantum integrals (see the contribution of Kellendonk, Kutz and Seiler; Chapter 10).

In recent years, there has been considerable progress in the theory of discrete integrable models, both classical and quantum mechanical. In particular, a connection between discrete integrable systems and the geometry of polyhedral surfaces and discrete curves has been discovered. It turns out that basic concepts of smooth geometry have natural discrete counterparts. In some sense, discrete integrable models appear to be even more fundamental then their smooth counterparts. This is reminiscent of classical and quantum mechanics, where quite similarly the newer theory—containing the older as a limiting case—is more fundamental than its predecessor.

The basic aspect of all models discussed in this book is their integrability in the sense of Hamiltonian systems. Recall that for continuous Hamiltonian systems on a $2n$-dimensional symplectic manifold called phase space, trajectories of the evolution are flow lines of a Hamiltonian vector field, generated by a function on phase space, the Hamiltonian. The evolution of a discrete Hamiltonian system on such a space is given by a symplectomorphism: a map which is diffeomorphic and preserves symplectic structure. A Hamiltonian system—continuous or discrete—is called integrable if there exist n independent integrals of motion (functions which are constant on trajectories of the evolution, i.e. which commute with respect to the Poisson bracket). A differential or a difference equation is called integrable if it describes the evolution of an integrable Hamiltonian system. The sine-Gordon equation is an example. It is, however, an infinite-dimensional Hamiltonian system, and this adds some analytic complications. To avoid these, some authors refer to integrability in a looser sense, namely, requiring only that there are infinitely many independent integrals of motion.

The simplest example for a discrete Hamiltonian evolution is the restriction of a Hamiltonian flow to some integer points in time. In this case, one says that the symplectomorphism has an interpolating flow. Not all symplectomorphisms are of this nature, and even worse, not all symplectomorphisms have a single integral in terms of a continuous function on phase space. A systematic procedure to construct integrable models with an interpolating flow based on the so called R-matrix approach is explained in the contribution of Suris (Chapter 7).

As we already mentioned, the sine-Gordon equation (0.1) is one of the simplest integrable partial differential equations. It will serve as a model for explaining some of the main concepts of this book.

The sine-Gordon equation describes surfaces with constant negative Gaussian curvature (K-surface). More precisely, $\phi(x, y)$ is the angle between the asymptotic lines on an asymptotically parameterized K-surface $(x, y) \mapsto F(x, y) \in \mathbb{R}^3$. Discrete K-surfaces are maps $F : \mathbb{Z}^2 \to \mathbb{R}^3$ with special natural geometric properties. Every image point together with all its nearest neighbors belongs to a single plane. Furthermore, the Euclidean distance between neighboring image points in each of the two coordinate directions is constant. Discrete K-surfaces are parameterized by the second-order difference equation (for notations and more details see the contribution of Bobenko and Pinkall; Chapter 1) given by

$$\phi_u + \phi_d - \phi_l - \phi_r = 2\arg(1 + ke^{-i\phi_l}) + 2\arg(1 + ke^{-i\phi_r}),$$

or in exponential ($z = e^{i\phi}$) form

$$z_u z_d = \frac{z_l + k}{1 + kz_l}\frac{z_r + k}{1 + kz_r}.$$

This equation is integrable and posesses causal propagation. Similar to the smooth case, ϕ is the angle between neighboring edges. k is a discretization parameter.

The quantum discrete sine-Gordon equation is

$$Q_u Q_d = \frac{e^{ih}Q_l + k}{1 + ke^{ih}Q_l}\ \frac{e^{ih}Q_r + k}{1 + ke^{ih}Q_r}.$$

Here the fields Q_j, $j = u, d, l, r$, are unitary operators, which do not commute any more, as opposed to the classical case. In the context of a discrete quantum field theory, the discrete sine-Gordon equation defines an automorphism of the observable algebra which is a non-commutative torus of arbitrary dimension. It is interpreted as discrete time evolution. Actually, the commutation rules are such that among the operators Q on a space-like staircase line only the nearest neighbors do not commute:

$$Q_d Q_l = e^{-2ih} Q_l Q_d, \qquad Q_d Q_r = e^{-2ih} Q_r Q_d.$$

Notice the similarity of the quantum and classical equations.

The simplest special case of the sine-Gordon model when $\phi(x,y)$ depends on the combination $t = x + y$ only is the pendulum equation

$$\phi_{tt} + \sin\phi = 0. \tag{0.4}$$

Let us describe the corresponding discrete model in some detail since it serves well for explaining some of the general concepts used throughout this text (for details see the contribution by Kellendonk, Kutz and Seiler; Chapter 10). The phase space of the discrete pendulum is the two-dimensional torus $M = S^1 \times S^1$ with coordinate functions z_1 and z_2, which are complex numbers of modulus one. The Poisson structure on M is given by the equation,

$$\{z_1, z_2\} = z_1 z_2$$

The equation of motion $(z_1, z_2) \mapsto (z_2, z_3)$ is a special case ($z_l = z_r$) of the discrete sine-Gordon equation[2] and is given by the formula

$$z_1 z_3 = \left(\frac{k + z_2}{1 + kz_2}\right)^2,$$

[2]For simplicity, we refer to a special case of the discrete pendulum.

which is a discretization of (0.4). The dynamics preserves the Poissonian structure and the level sets of the function

$$H(z_1, z_2) = 2(z_1 + z_2 + z_1^{-1} + z_2^{-1}) + k(z_1 z_2 + z_1^{-1} z_2^{-1}) + k^{-1}(z_1 z_2^{-1} + z_2 z_1^{-1}).$$

Hence the discrete pendulum is integrable. This is a remarkable fact, since the discrete pendulum equation is rather close to the standard map, which is the paradigm of chaotic and therefore non-integrable dynamics, after all.

The algebra of observables of the discrete quantum pendulum is known under three different names. It is called the two-dimensional quantized torus, the rotation algebra and the discrete Weyl–Heisenberg algebra. The latter is generated by two unitaries Q_1 and Q_2, which satisfy the commutation relation

$$Q_1 Q_2 = e^{-2ih} Q_2 Q_1.$$

Quantum dynamics $(Q_1, Q_2) \mapsto (Q_2, Q_3)$ is by definition derived from classical dynamics in the most naive manner:

$$Q_3 Q_1 = \left(\frac{k + e^{ih} Q_2}{1 + k e^{ih} Q_2} \right)^2. \qquad (0.5)$$

This is interpreted as the quantum discrete time evolution and notably an algebra automorphism. In close analogy to the classical situation the operator

$$H(Q_1, Q_2) = 2(Q_1 + Q_2 + Q_1^* + Q_2^*) + \\ k(e^{ih} Q_1 Q_2 + e^{-ih} Q_2^* Q_1^*) + k^{-1}(e^{ih} Q_2 Q_1^* + e^{-ih} Q_1 Q_2^*)$$

is a quantum integral, i.e., it is invariant under the automorphism defined by equation (0.5). Such operators are known as operators of the Hofstadter type, which play an important role in solid state physics.

In Fig. 1 the interrelations between geometry, classical Hamiltonian dynamics—in short: classical systems—and quantum physics in their continuous and discrete versions are shown schematically. Looking at two neighboring boxes, it is straightforward to go in one direction by taking an appropriate limit or by simple computation. The other direction is typically much more difficult, because it involves some clever guess. To go, for instance, from a box in the right column to the one adjacent to the left, one takes the limit of zero lattice spacing. Similarly, the classical limit leads from non-commutative quantum theory to commutative classical theory. To go from geometry to classical Hamiltonian systems one simply analyzes the algebraic implications of the consistency equations, the Lax representation. The arrows are oriented in the simple direction.

The other directions are called deformations. Discrete geometry and quantum physics are an example of this type, both emanating from the same classical commutative theory. The key problem in all kinds of deformation theory is the preservation of their internal structure. In the present case, such structures are

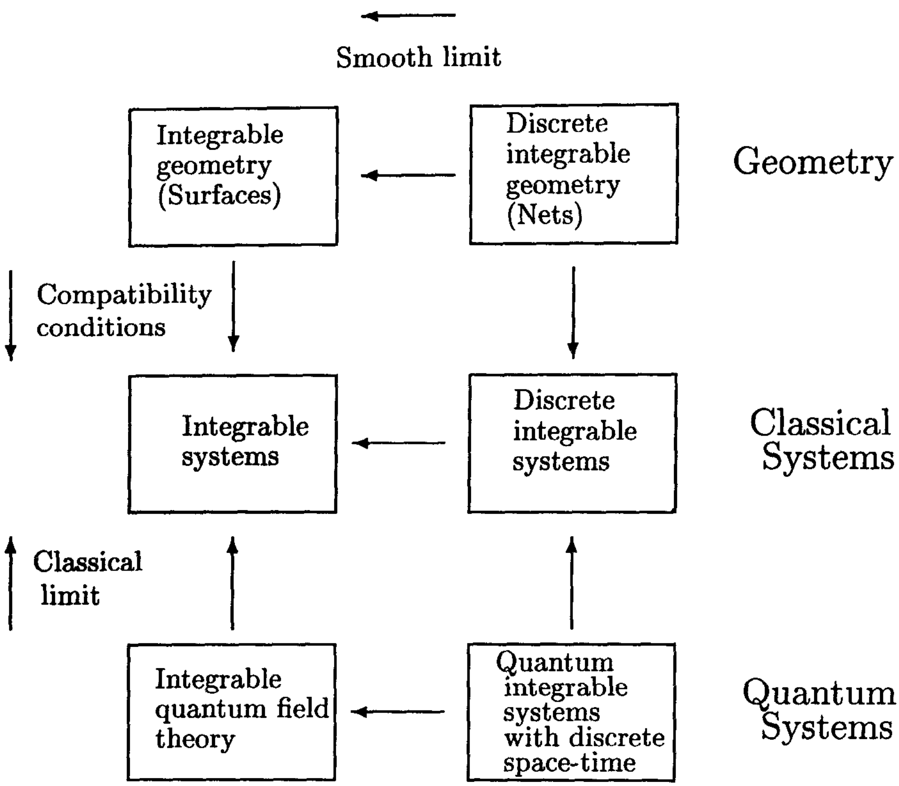

FIG. 1. Integrable geometry and physics

integrability, intrinsic geometric properties, and what is called the correspondence principle of quantum mechanics.

The structure of the book naturally follows Fig. 1 — the contributions are organized in three sections: Geometry, Classical Systems, and Quantum Systems. Most of the contributions address the following problem: how to invert the horizontal arrows of the diagram, i.e. how to find discretizations preserving the integrability property. In particular, Bobenko and Pinkall (Chapter 1) introduce special nets which are discrete analogs of the classical geometries. Suris (Chapter 7) develops a discretization method based on the R-matrix description. A different method, which is useful also in other contexts, is based on the discretization of the action. This is the main idea in the short note by Kutz (Chapter 9). Faddeev and Volkov (Chapter 11) describe quantum integrable models on discrete space–time concentrating on the algebraic part of the theory.

Of course, the real picture of interrelations is more complicated than the present diagram suggests. In particular, geometry and quantum physics are not just two independent extensions of classical theory. There are several direct connections between them. Here one can mention, for example, an important application of the quantum theory for constructing topological invariants (see the contribution by Kashaev; Chapter 13). In some cases, for example for the discrete sine-Gordon equation, the introduction of the proper quantum variables has been motivated geometrically (see the contribution by Kellendonk, Kutz and Seiler; Chapter 10).

In a way, the first row of the diagram represents the essence of the theory of solitons in the smooth setting: given a system, first find the corresponding Lax representation, and secondly find the Hamiltonian interpretation (R-matrix description). The latter is a method to find a quantum integrable version of the model.

Let us now give a short description of the contents of different chapters of the book.

Part I: Geometry

A.I. Bobenko and **U. Pinkall** start this section with definitions of discrete analogs of various classes of surfaces and mappings described by integrable systems. These discretizations are characterized by the property that the integrability is preserved, i.e., they are described by discrete integrable systems. As a corollary, rich algebraic structures such as the loop group description, the Bäcklund–Darboux transformation, etc. of the corresponding smooth geometries persist in the discrete case. The contribution is a survey of results obtained in this field, and it serves as an introduction to other contributions in that chapter. Combining methods of soliton theory with geometrical intuition, many concrete discrete geometries are described.

Two basic examples are:

- discrete K-surfaces (surfaces with constant negative Gaussian curvature),
- discrete H-surfaces (surfaces with constant mean curvature),

which are considered in detail. These nets are first derived analytically by discretizing the Lax representation of the corresponding smooth surfaces, preserving the corresponding loop groups. After that, geometrical properties of the nets defined in this way are studied. It is shown that these are natural discrete analogs of geometric properties of the corresponding smooth surfaces.

Other discrete integrable geometries are obtained from generalizations or specializations of these two examples. In all the cases geometrical definitions of the nets are presented. These definitions refer neither to integrable systems nor to the loop group interpretation. For some cases the corresponding Cauchy problems and examples of surfaces are discussed.

U. Hertrich-Jeromin, T. Hoffmann, and **U. Pinkall** continue with a purely geometrical description of a discrete version of the Darboux transform for isothermic surfaces. Christoffel and Darboux transforms of discrete isothermic nets in the four-dimensional Euclidean case are studied. Definitions and basic properties are derived. Analogies with the smooth case are discussed and discrete Ribaucour congruencies are defined. Surfaces of constant mean curvature are special among all isothermic surfaces: they can be characterized by the fact that their parallel constant mean curvature surfaces are Christoffel and Darboux transforms at the same time. This characterization is used as a definition of discrete nets of constant mean curvature. Starting with this definition, basic properties of discrete nets of constant mean curvature are derived.

This chapter can be considered as complementary to the contribution of Bobenko and Pinkall, since it does not refer to the analytic description, loop groups and theory of discrete integrable systems. The discrete isothermic and constant mean curvature nets are investigated completely in internal geometrical terms.

Next **T. Hoffmann** describes discrete Amsler surfaces and shows how they are related to a discrete Painlevé III equation. Amsler surfaces are surfaces with constant negative Gaussian curvature which have two straight asymptotic lines. The corresponding solution of the sine-Gordon equation is known to reduce to a special case of the Painlevé III equation. On the other hand, the Painlevé III equation can be obtained as an isomonodromy condition for the moving frame of the surface with respect to the spectral parameter. It is shown that both these properties persist for discrete surfaces with constant negative Gaussian curvature (discrete K-surfaces). Starting from the geometric properties of the Amsler surfaces, a discrete analogue of the surface is obtained. A discrete Painlevé III equation is derived as an isomonodromy problem for the extended (discrete) frame.

In his next contribution **T. Hoffmann** presents a discrete analogue of the Dorfmeister–Pedit–Wu (DPW) method. He shows that discrete H-surfaces and the corresponding solutions of the discrete sinh-Gordon equation can be constructed from discrete holomorphic maps and that the discrete DPW method is a quite efficient way of construction. As an example, discrete Delauney and Smyth surfaces are constructed which correspond to pendulum and rotational invariant solutions of the (discrete) sinh-Gordon equation.

Affine spheres are another well-known example of integrable geometry. **A.I. Bobenko** and **W.K. Schief** start with a geometrically motivated definition of discrete indefinite affine spheres. These nets are defined as distinguished discrete A-surfaces characterized by a certain affine property of elementary quadrilaterals. The geometrical definition is a starting point of further algebraic investigation. It is shown that the corresponding difference frame equations by inserting of an additional parameter can be extended to the Lax representation. The corresponding difference Gauss equation is derived. It is an integrable discretization of the Tzitzeica equation. Discrete indefinite affine spheres are interpreted in terms of the permutability theorem for the Bäcklund transformations of the smooth affine spheres. A Bäcklund transformation generating discrete affine spheres is obtained in a purely algebraic manner. An interpretation of the Gauss equations for discrete affine spheres in terms of loop groups is given.

This part closes with a contribution by **A. Doliwa** and **P.M. Santini**, describing integrable features of geometry of discrete curves and higher dimensional lattices. A classical result in soliton theory establishes the equivalence between the nonlinear Schrödinger dynamics, the evolution of the Heisenberg ferromagnet, and the motion of a vortex filament under the Localized Induction Approximation. This approach is generalized to a discrete context by considering the evolution of a discrete (piecewise linear) curve whose Frenet equation is the Ablowitz–Ladik spectral problem. A natural geometrical analog of the Laplace sequence for conjugate nets (elementary quadrilaterals are planar) is suggested. It is shown that algebraically it is described by the discrete generalized Toda system. Multidimensional lattices with planar quadrilaterals (discrete conjugate systems) and their special case–circular lattices (O-systems)–are also studied.

Part II: Classical Systems

First, **Y.B. Suris** shows how to define and study integrable discretizations using the r-matrix approach. A short overview of the Lie-algebraic r-matrix scheme for finite-dimensional integrable systems of classical mechanics is given. A general abstract approach to the bi-Hamiltonian property of such systems is discussed, based on the concept of linear and quadratic r-matrix Poisson brackets on Lie algebras and associative algebras. Lax equations are identified as (bi)-Hamiltonian systems on Lie algebras corresponding to Ad^*-invariant Hamiltonian functions. A connection between Lax representations and factorization problems in Lie groups is recalled.

A general formula is obtained defining a map as time $-h$ shift along the trajectories of an arbitrary Hamiltonian flow in r-matrix hierarchies. This formula is the basis of a systematic derivation of integrable discretizations for such flows. A characteristic feature of the discretizations obtained along these lines is that they belong to the same hierarchy as their continuous time counterparts, i.e. they have the same Lax matrices and the same integrals of motion. An underlying invariant Poisson structure, interpolating Hamiltonian flow, and a solution in terms of matrix factorizations appear as by-products of the discretizations obtained by this approach.

For a few years there have existed "exact" discretizations of the famous Painlevé equations, PI-VI. They have been obtained by various methods. A striking feature is the fact that there seems to be more than one discrete analog corresponding to one and the same continuous Painlevé transcendent. In his contribution **Frank Nijhoff** discusses a particular method to obtain such discrete Painlevé equations—the lattice similarity approach by which one applies a proper analogue of similarity reduction to integrable lattice equations such as the lattice KdV and modified KdV equation. This approach could ultimately explain the structure behind the discrete Painlevé equations and reveal some of the relations between the various alternate forms of the discrete transcendents. One particular new class of equations that comes out of this approach is a class of third-order difference equations of "Schwarzian" type related to the second-order difference Painlevé equations.

Nadja Kutz finishes this part with an investigation of Lagrangian systems which belong to evolutions of doubly discrete sine-Gordon type. The phase space structure of these models and the relations between them are discussed in terms of symplectic geometry.

Part III: Quantum Systems

J. Kellendonk, N. Kutz, and **R. Seiler** start this part with a discussion of some discrete Schrödinger operators. These operators arise as Hamiltonians for the description of charged particles in a magnetic field. The paradigm is the Hofstadter Hamiltonian. Miraculously, these operators are at the same time integrals for the quantum pendulum. They all belong to the discrete Weyl–Heisenberg algebra on $\mathbb{Z}^2$ or, in another terminology, to the rotation algebra or the quantized torus.

The first part of the chapter is about the analysis of the discrete Weyl–Heisenberg algebra, its irreducible representation, and automorphism. This is a basic concept, which is used in one way or other in every contribution in this section. The discrete Weyl–Heisenberg algebra can be interpreted as the kinematical framework of discrete elementary quantum mechanics. In the following, models of the Hofstadter type and the integrals of the quantum pendulum are explained and their spectrum is analyzed by means of the Bethe Ansatz introduced into this context by Wiegmann and Zabrodin, and Faddeev and Kashaev. In particular this applies also in the case of irrational flux.

L. Faddeev and **A. Volkov** continue with developing the algebraic framework of an integrable discrete quantum field theory in one space and one time dimension using the example of the Liouville model. The explicit form of the time-one (discrete) evolution operator is given for several examples of integrable models on 1+1-dimensional discrete space–time. The locality of the Hamiltonian in classical continuous limit is changed into the multiplicative locality of the evolution operator. The exchange relations for the corresponding local factors are discussed.

Next **R. Kashaev** and **N. Reshetikhin** study the affine Toda field theory as a 2+1-dimensional system. The third dimension appears as the discrete space dimension, corresponding to the simple roots in the A_N affine root system, enumerated according to the cyclic order on the A_N affine Dynkin diagram. They show that there exists a natural discretization of the affine Toda theory, where the equations of motion are invariant with respect to permutations of all discrete coordinates. The discrete evolution operator is constructed explicitly. The thermodynamic Bethe Ansatz of the affine Toda system is studied in the limit $L, N \to \infty$. Some conjectures about the structure of the spectrum of the corresponding discrete models are stated.

The contribution of **R. Kashaev** is a survey of recent results on quantum invariants of knots and links, associated with the quantum dilogarithm at roots of unity. The construction can be considered as a combinatorial (simplicial) analog of Witten's realization of the Jones polynomial as expectation values of Wilson lines in quantum Chern–Simons theory. In the classical limit the invariant obtained, evaluated on hyperbolic knots and links, seems to reproduce their hyperbolic volumes. The latter implies the possible relation of the construction to quantum Euclidean three-dimensional gravity with negative cosmological constant.

T. Richter and **R. Seiler** finish this section with a short presentation of a model for the quantum Hall effect in the framework of discrete elementary quantum mechanics.

Discrete integrable geometry and quantum physics is by no means a finished piece of mathematical science. On the contrary, it seems to be at its beginning. The goal of research in this field, as we see it, is to develop a theory to complete Fig. 1. The continuum theory—geometry, classical and quantum integrable models—should be looked at as just a limiting object of this more fundamental discrete theory. The modern state of research in this field presented in this book shows that many fragments of this discrete master theory are already constructed. On the other hand, some important ingredients of the theory are still missing. We believe that there will be further rapid progress in the field and that in the near future all the ingredients of Fig. 1 will be developed to the same extent as the theories of the left (continuous) part of the figure are nowadays.

We are grateful to Ulrich Pinkall who initiated research in the field on the border of geometry and discrete integrable systems and whose ideas played a crucial role for the success of the project. We would like to thank Tim Hoffmann for his wonderful work with LaTeX, and helping us to produce a homogeneous book. Finally we are grateful to the Deutsche Forschungsgemeinschaft for financial support of the SFB 288 and to the Erwin Schrödinger Institute for supporting the conference in Vienna which inspired this book.

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