Let Us Use White Noise

$$\int_{\mathcal{N}'} \exp\left(i\langle x, f\rangle\right) d\mu(x) = \exp\left(-\frac{1}{2}|f|^2\right)$$

T Hida • L Streit
Editors

World Scientific

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Editors

T Hida

Nagoya University, Japan & Meijo University, Japan

L Streit

University of Bielefeld, Germany & University of Madeira, Portugal

World Scientific

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Preface

Why Should We?
Some Personal Comments from One Happy User.

When I embarked into the world of mathematical physics, learning about "axiomatic" quantum field theory from H. Lehmann and W. Zimmermann, and reading Borchers, Symanzik, Haag, Streater, and Wightman, I was impressed with the beauty and clarity of the LSZ and Wightman frameworks, - and quite depressed afterwards. In his book on the general theory of quantized fields, the great Res Jost wrote at the time:[1] "We had very compelling reasons for not mentioning any models except free fields. No interesting models are known ...", bad news for a junior researcher who wondered: "Will there ever be any?" And if so: "How to construct them?" Of course there were attempts; the best of them were visionary - and bad mathematics. Let me single out Feynman's "sum over histories" and the observation of Coester and Haag[2] that quantum field theory dynamics is in fact encoded in the vacuum. We knew even then that the Feynman integral was not an integral, and that the manipulations of Coester and Haag could not be justified mathematically, but it also became quite clear that there was by far not enough of infinite dimensional analysis in the physicists' mathematical tool kit. Things did get better with the physics breakthrough that goes under the name of "constructive quantum field theory", and when, on the mathematical side, 40 years ago the foundations of white noise analysis were laid.

Of course white noise analysis does not claim a monopoly: Mallavin calculus is a close relative, much like in finite dimensional analysis where there are many different Gelfand triples, suited to address particular needs. As Paul André Meyer once said in a heated debate - don't argue about the advantages of one approach or the other, show what you can do with the one that you prefer. My good friend J-A. Yan, together with Z-Y. Huang, presents the two approaches side by side in his beautiful book.[3]

So why should we use white noise analysis? Well one reason is of course that it fills that earlier gap in the tool kit. As Hida would put it, white noise provides us with a useful set of *independent* coordinates, parametrized by "time". And there is a feature which makes white noise analysis extremely user-friendly. Typically the physicist — and not only he — sits there with some heuristic ansatz, like e.g. the famous Feynman "integral", wondering whether and how this might make sense mathematically. In many cases the characterization theorem of white noise analysis provides the user with a sweet and easy answer. Feynman's "integral" can now be understood, the ansatz of Haag and Coester is now making sense via Dirichlet forms, and so on in many fields of application. There is mathematical finance, there have been applications in biology, and engineering,[4] many more than we could collect in the present volume, for some of them see e.g. Bernido and Bernido.[5]

Finally, there is one extra benefit: when we internalize the structures of Gaussian white noise analysis we will be ready to meet another close relative — we will enjoy the important similarities and differences which we encounter in the Poisson case, championed in particular by Y. Kondratiev and his group, let us look forward to a companion volume on the uses of Poisson white noise.

The present volume is essentially a collection of autonomous contributions. Fortunately however, the introductory chapter on white noise analysis was made available to the other authors early on for reference and to facilitate their efforts towards conceptual and notational coherence.

At the end of such a preface one has the right of a note of gratitude to friends and teachers. Some of the latter I have already mentioned. Then there is the "white noise community", too big by now to list it here. But I guess I have made the acquaintance and won the friendship of almost all of the white noise mathematicians you find quoted in the present volume. I also thank all of them for what they taught me. I thank the authors of the different chapters, I thank S. C. Lim of World Scientific for his invitation, help, and great patience, and can now finally, with the contributions in hand, enjoy the encouragement I got for this undertaking. May the readers enjoy those contributions and may they feel encouraged to use white noise.

Ludwig Streit, August 2016

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2. Fritz Coester and Rudolf Haag: Representation of States in a Field Theory with Canonical Variables. Phys. Rev. 117, 1137 (1960).
3. Zhi-Yuan Huang, Jia-An Yan; Introduction to Infinite Dimensional Stochastic Analysis. Springer, 2000.
4. I am thinking particular of the work of Roger Ghanem who struggled, unfortunately in vain, to meet the deadlines of this book.
5. Christopher and Victoria Bernido: Methods And Applications Of White Noise Analysis In Interdisciplinary Sciences. World Scientific, 2015.

Contents

Chapter 1

White Noise Analysis: An Introduction

Maria João Oliveira
Universidade Aberta, P 1269-001 Lisbon, Portugal
CMAF-CIO, University of Lisbon, P 1749-016 Lisbon, Portugal
mjoliveira@ciencias.ulisboa.pt

The starting point of White Noise Analysis[11] and[2,14–16,20,21,34,39] is a real separable Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$ and the corresponding norm $|\cdot|$, and a nuclear triple

$$\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}',$$

where $\mathcal{N}$ is a nuclear space densely and continuously embedded in $\mathcal{H}$. Of course, in a general framework, *a priori* there are several different possible nuclear spaces. However, in concrete applications, the application will suggest the use of particular nuclear triples. For example, in the study of intersection local times of d-dimensional Brownian motions it is natural to consider the space $\mathcal{H} = L^2(\mathbb{R},\mathbb{R}^d) =: L^2_d(\mathbb{R})$ of all vector valued square integrable functions with respect to the Lebesgue measure on $\mathbb{R}$ and the Schwartz space $\mathcal{N} = S(\mathbb{R},\mathbb{R}^d) =: S_d(\mathbb{R})$ of vector valued test functions, while in the treatment of Feynman integrals the spaces $L^2(\mathbb{R}) := L^2(\mathbb{R},\mathbb{R})$, $S(\mathbb{R}) := S(\mathbb{R},\mathbb{R})$ are the natural ones.

Since nuclear triples are the basis of the whole White Noise Analysis, we start by briefly recalling the main background of the theory of nuclear spaces, due to A. Grothendieck.[7] For simplicity, instead of general nuclear spaces, cf. e.g.,[40,42,45,50] we just consider nuclear Fréchet spaces, which are the only ones needed in this book. For more details and the proofs see e.g.[2,3,9,14].

1. Nuclear Triples

As before, let $\mathcal{H}$ be a real separable Hilbert space. We consider a family of real separable Hilbert spaces $\mathcal{H}_p$, $p \in \mathbb{N}$, with Hilbertian norm $|\cdot|_p$ such that

$$\mathcal{H} \supset \mathcal{H}_1 \supset \ldots \supset \mathcal{H}_p \supset \ldots$$

so that the corresponding system of norms is ordered:

$$|\cdot| \leq |\cdot|_1 \leq \ldots \leq |\cdot|_p \leq \ldots.$$

In addition, we assume that the intersection of the Hilbert spaces $\mathcal{H}_p$, denoted by

$$\mathcal{N} := \bigcap_{p\in\mathbb{N}} \mathcal{H}_p, \tag{1}$$

is dense in each space $\mathcal{H}_p$, $p \in \mathbb{N}$.

Definition 1. The linear space $\mathcal{N}$ is called nuclear whenever for every $p \in \mathbb{N}$ there is a $q > p$ such that the canonical embedding $\mathcal{H}_q \hookrightarrow \mathcal{H}_p$ is a Hilbert-Schmidt operator.

From now on we shall assume that all spaces (1) are nuclear and fix on $\mathcal{N}$ the *projective limit topology*, that is, the coarsest topology on $\mathcal{N}$ with respect to which all canonical embeddings $\mathcal{N} \hookrightarrow \mathcal{H}_p$, $p \in \mathbb{N}$, are continuous. Or, in an equivalent way, a sequence $(\xi_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{N}$ converges to $\xi \in \mathcal{N}$ if and only if $(\xi_n)_{n\in\mathbb{N}}$ converges to ξ in every Hilbert space $\mathcal{H}_p$, $p \in \mathbb{N}$. It turns out that a nuclear space $\mathcal{N}$ endowed with the projective limit topology is a complete metrizable locally convex space, meaning that it is a Fréchet space. In order to mention explicitly this topology fixed on $\mathcal{N}$, we shall use the notation

$$\mathcal{N} = \underset{p\in\mathbb{N}}{\mathrm{prlim}}\, \mathcal{H}_p$$

and call such a topological space a *projective limit* or a *countable limit of the family* $(\mathcal{H}_p)_{p\in\mathbb{N}}$.

For each $p \in \mathbb{N}$, let now $\mathcal{H}_{-p}$ be the Hilbertian dual space of $\mathcal{H}_p$ with respect to $\mathcal{H}$ with the corresponding Hilbertian norm $|\cdot|_{-p}$. By the general duality theory cf. e.g.,[9] we have

$$\mathcal{N}' = \bigcup_{p\in\mathbb{N}} \mathcal{H}_{-p},$$

where $\mathcal{N}'$ is the dual space of $\mathcal{N}$ with respect to $\mathcal{H}$. Unless stated otherwise, we shall consider $\mathcal{N}'$ endowed with the inductive limit topology, that is, the finest topology on $\mathcal{N}'$ with respect to which all embeddings $\mathcal{H}_{-p} \hookrightarrow \mathcal{N}'$ are continuous. As a topological space, we shall denote it by

$$\mathcal{N}' = \underset{p\in\mathbb{N}}{\mathrm{indlim}}\, \mathcal{H}_{-p}$$

and call it an *inductive limit* of the family $(\mathcal{H}_{-p})_{p\in\mathbb{N}}$.

In this way, using the Riesz representation theorem to identify $\mathcal{H}$ with its dual space $\mathcal{H}'$, we have defined a so-called *nuclear* or *Gelfand triple*:

$$\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'.$$

By construction, it turns out that the bilinear dual pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{N}'$ and $\mathcal{N}$ is defined as an extension of the inner product on $\mathcal{H}$:

$$\langle h, \xi \rangle = (h, \xi), \quad h \in \mathcal{H}, \xi \in \mathcal{N}.$$

Example 1. (i) The Schwartz space $S(\mathbb{R})$ of rapidly decreasing C^∞-functions on $\mathbb{R}$ endowed with its usual topology given by the system of seminorms

$$\sup_{u\in\mathbb{R}} \left| u^n \frac{d^m \xi}{du^m}(u) \right|, \quad \xi \in S(\mathbb{R}), m, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$$

is a first example of a nuclear space. Indeed, given the Hamiltonian of the quantum harmonic oscillator, that is, the self-adjoint operator on $L^2(\mathbb{R})$ defined on $S(\mathbb{R})$ by

$$(H\xi)(u) := -\frac{d^2\xi}{du^2}(u) + (u^2 + 1)\xi(u), \quad u \in \mathbb{R},$$

we can define a system of norms $|\cdot|_p$ by setting

$$|\xi|_p := |H^p \xi|, \quad \xi \in S(\mathbb{R}), p \in \mathbb{N},$$

where the last norm is the one on $L^2(\mathbb{R})$. It turns out (cf. e.g.,[12,43,47]) that this system of norms is equivalent to the initial system of seminorms, and thus both systems lead to equivalent topologies on $S(\mathbb{R})$. In addition, the completion of $S(\mathbb{R})$ with respect to each norm $|\cdot|_p$ yields a family of Hilbert spaces $\mathcal{H}_p$ and

$$S(\mathbb{R}) = \operatorname*{prlim}_{p\in\mathbb{N}} \mathcal{H}_p,$$

see e.g.,[14]. Therefore, for the dual space $S'(\mathbb{R})$ of $S(\mathbb{R})$ (with respect to $L^2(\mathbb{R})$) of Schwartz tempered distributions we have

$$S'(\mathbb{R}) = \operatorname*{indlim}_{p\in\mathbb{N}} \mathcal{H}_{-p}.$$

(ii) The previous example extends to the space $S_d(\mathbb{R})$ of vector valued Schwartz test functions for the operator H defined on $S_d(\mathbb{R})$ by

$$(H\boldsymbol{\xi})(u) := ((H\boldsymbol{\xi})_1(u), \ldots, (H\boldsymbol{\xi})_d(u)), \quad \boldsymbol{\xi} = (\xi_1, \ldots, \xi_d), \xi_i \in S(\mathbb{R}) \quad (2)$$

with

$$(H\boldsymbol{\xi})_i(u) := -\frac{d^2\xi_i}{du^2}(u) + (u^2+1)\xi_i(u) = (H\xi_i)(u), \quad i=1,\dots,d, u \in \mathbb{R}.$$

This leads to the following system of increasing Hilbertian norms $|\cdot|_p$, $p \in \mathbb{N}$,

$$|\boldsymbol{\xi}|_p^2 := \sum_{i=1}^d |\xi_i|_p^2 = \sum_{i=1}^d |H^p\xi_i|^2,\ \boldsymbol{\xi} = (\xi_1,\dots,\xi_d), \xi_i \in S(\mathbb{R}), i=1,\dots,d, \quad (3)$$

where the last sum in (3) is the square of the $L^2_d(\mathbb{R})$-norm of (2), and to the corresponding Hilbert spaces $\mathcal{H}_p$ defined by completion of $S_d(\mathbb{R})$ with respect to the norms (3). As in (i), we have

$$S_d(\mathbb{R}) = \operatorname*{prlim}_{p\in\mathbb{N}} \mathcal{H}_p, \quad S'_d(\mathbb{R}) = \operatorname*{indlim}_{p\in\mathbb{N}} \mathcal{H}_{-p},$$

being $S'_d(\mathbb{R})$ the space of vector valued Schwartz tempered distributions.

(iii) Example (i) also extends to the Schwartz space $S(\mathbb{R}^d, \mathbb{R})$ of smooth functions on $\mathbb{R}^d$, $d \geq 2$, of rapid decrease (shortly $S(\mathbb{R}^d)$) and to its dual space $S'(\mathbb{R}^d)$ of Schwartz tempered distributions. In this case, the usual topology on $S(\mathbb{R}^d)$ is given by the family of seminorms indexed by multi-indices $(\alpha_1,\dots,\alpha_d)$, $(\beta_1,\dots,\beta_d)$ in $\mathbb{N}_0^d$,

$$\sup_{\mathbf{u}=(u_1,\dots,u_d)\in\mathbb{R}^d} \left| u_1^{\alpha_1}\dots u_d^{\alpha_d} \left(\partial_1^{\beta_1}\dots\partial_d^{\beta_d}\xi\right)(\mathbf{u})\right|, \quad \xi \in S(\mathbb{R}^d),$$

where ∂_i, $i=1,\dots,d$, is the partial derivative on $\mathbb{R}^d$ with respect to the i-th coordinate. Given the Hamiltonian of the quantum harmonic oscillator, that is, the self-adjoint operator on $L^2(\mathbb{R}^d,\mathbb{R}) =: L^2(\mathbb{R}^d)$ defined on $S(\mathbb{R}^d)$ by

$$(H\xi)(\mathbf{u}) := -(\Delta\xi)(\mathbf{u}) + (|\mathbf{u}|^2+1)\xi(\mathbf{u}), \quad \mathbf{u} \in \mathbb{R}^d,$$

being Δ the Laplacian on $\mathbb{R}^d$, we define a system of norms $|\cdot|_p$ on $S(\mathbb{R}^d)$ by

$$|\xi|_p := |H^p\xi|, \quad \xi \in S(\mathbb{R}^d), p \in \mathbb{N},$$

where the last norm is the one on $L^2(\mathbb{R}^d)$. As in Example (i), it turns out cf. e.g.,[12,43,47] that such a system is equivalent to the above system of seminorms, leading then to equivalent topologies on $S(\mathbb{R}^d)$. In addition, cf. e.g.,[14] we have

$$S(\mathbb{R}^d) = \operatorname*{prlim}_{p\in\mathbb{N}} \mathcal{H}_p,$$

where each $\mathcal{H}_p$, $p \in \mathbb{N}$, is the Hilbert space obtained by completion of $S(\mathbb{R}^d)$ with respect to the norm $|\cdot|_p$. Thus

$$S'(\mathbb{R}^d) = \operatorname*{indlim}_{p\in\mathbb{N}} \mathcal{H}_{-p}.$$

2. Gaussian Space

Given a nuclear triple $\mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'$, let $\mathcal{C}_\sigma(\mathcal{N}')$ be the σ-algebra on $\mathcal{N}'$ generated by the *cylinder sets*

$$\{x \in \mathcal{N}' : (\langle x, \varphi_1 \rangle, \dots, \langle x, \varphi_n \rangle) \in B, \varphi_1, \dots, \varphi_n \in \mathcal{N}, B \in \mathcal{B}(\mathbb{R}^n), n \in \mathbb{N}\},$$

where $\mathcal{B}(\mathbb{R}^n)$, $n \in \mathbb{N}$, is the Borel σ-algebra on $\mathbb{R}^n$.

Theorem 1. ***(The Minlos Theorem*[37]*)*** *Let C be a complex-valued function on $\mathcal{N}$ fulfilling the following three properties:*

(i) $C(0) = 1$,
(ii) C *is continuous on* $\mathcal{N}$,
(iii) C *is positive definite, i.e.,*

$$\sum_{i,j=1}^{n} C(\xi_i - \xi_j) z_i \overline{z_j} \geq 0, \quad \xi_1, \dots, \xi_n \in \mathcal{N}, z_1, \dots, z_n \in \mathbb{C}, n \in \mathbb{N}.$$

Then, there is a unique probability measure μ_C on $(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'))$ which characteristic function is equal to C, that is, for all $\xi \in \mathcal{N}$

$$\int_{\mathcal{N}'} \exp\left(i\langle x, \xi \rangle\right) \, d\mu_C(x) = C(\xi). \tag{4}$$

For a presentation of the Minlos theorem, including support properties of the probability measure given by this theorem see.[10]

Remark 1. The analogous statement of the Minlos theorem for the nuclear space $\mathcal{N}$ replaced by the finite dimensional space $\mathbb{R}^d$ is the well-known Bochner theorem. Because of this, in the literature Theorem 1 is quite often called the Bochner-Minlos theorem as well.

Consider now the following particular positive definite continuous function defined on $\mathcal{N}$ by

$$C(\xi) = \exp\left(-\frac{1}{2}|\xi|^2\right), \quad \xi \in \mathcal{N}. \tag{5}$$

Then, by the Minlos theorem, we are given a (Gaussian) measure μ on $(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'))$ defined by (4) and (5).

Definition 2. We call the probability space $(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'), \mu)$ the Gaussian space associated with $\mathcal{N}$ and $\mathcal{H}$.

In particular, if $\mathcal{N} = S(\mathbb{R}^d)$ with the topology described in Example 1, the space $(S'(\mathbb{R}^d), \mathcal{C}_\sigma(S'(\mathbb{R}^d)), \mu)$ is called white noise with d-dimensional time parameter. If $d = 1$, we simply call it white noise.

Definition 3. For short we set

$$(L^2) := L^2(\mathcal{N}', \mathcal{C}_\sigma(\mathcal{N}'), \mu)$$

for the complex L^2 space.

In applications of White Noise Analysis, the space (L^2) plays an essential role. In order to distinguish clearly the inner product $(\cdot,\cdot)$ and the Hilbertian norm $|\cdot|$ on the real space $\mathcal{H}$ from those defined on the complex space (L^2), we shall denote the inner product on (L^2) by $(\!(\cdot,\cdot)\!)$ and the corresponding norm by $\|\cdot\|$. Furthermore, we shall assume that $(\!(\cdot,\cdot)\!)$ is linear in the first factor and antilinear in the second one, that is,

$$(\!(F_1, F_2)\!) := \int_{\mathcal{N}'} F_1(x)\bar{F}_2(x)\, d\mu(x), \quad F_1, F_2 \in (L^2),$$

where $\bar{F}_2$ is the complex conjugate function of F_2.

From the definition of the Gaussian measure μ given by (4) and (5), it follows straightforwardly that for every $\xi \in \mathcal{N}$, $\langle\cdot,\xi\rangle$ is a normally distributed random variable with variance $|\xi|^2$. Thus, for all $\xi \in \mathcal{N}$, $\xi \neq 0$,

$$\|\langle\cdot,\xi\rangle\|^2 = \int_{\mathcal{N}'} \langle x,\xi\rangle^2\, d\mu(x) = \frac{1}{\sqrt{2\pi|\xi|^2}} \int_{-\infty}^{+\infty} u^2 \exp\left(-\frac{u^2}{2|\xi|^2}\right) du = |\xi|^2.$$

Moreover, again by (4) and (5), the real process X defined on $\mathcal{N}' \times \mathcal{N}$ by $X_\xi(x) = \langle x,\xi\rangle$ is centered Gaussian with covariance

$$\int_{\mathcal{N}'} \langle x,\xi_1\rangle\langle x,\xi_2\rangle\, d\mu(x) = \frac{1}{2}\left(\|\langle\cdot,\xi_1+\xi_2\rangle\|^2 - \|\langle\cdot,\xi_1\rangle\|^2 - \|\langle\cdot,\xi_2\rangle\|^2\right) = (\xi_1,\xi_2).$$

As we have mentioned above, in this book we shall mostly choose $\mathcal{N}$ to be the Schwartz space $S(\mathbb{R}^d)$, $S_d(\mathbb{R})$, or $S(\mathbb{R})$ of test functions and $\mathcal{H}$ to be $L^2(\mathbb{R}^d)$, $L^2_d(\mathbb{R})$, or $L^2(\mathbb{R})$, respectively. In all these cases, $\mathcal{N}$ is dense in $\mathcal{H}$. This is an assumption fixed on general $\mathcal{N}$ and $\mathcal{H}$ from the very beginning. Therefore, the above considerations allow an extension of the mapping

$$\mathcal{N} \ni \xi \mapsto \langle\cdot,\xi\rangle \in (L^2)$$

to a bounded linear operator

$$\mathcal{H} \ni f \mapsto \langle\cdot,f\rangle \in (L^2)$$

defined at each $f \in \mathcal{H}$ by

$$\langle\cdot,f\rangle := (L^2) - \lim_n \langle\cdot,\xi_n\rangle,$$

where $(\xi_n)_{n\in\mathbb{N}}$ is any sequence in $\mathcal{N}$ converging to f in $\mathcal{H}$. Moreover, $\|\langle\cdot,f\rangle\| = |f|$ for all $f \in \mathcal{H}$.

Proposition 1 ([14]). *The process X defined on $\mathcal{N}' \times \mathcal{H}$ by $X_f(x) = \langle x, f\rangle$ is centered Gaussian with covariance*

$$(\!(\langle \cdot, f\rangle, \langle \cdot, g\rangle)\!) = \int_{\mathcal{N}'} \langle x, f\rangle \langle x, g\rangle \, d\mu(x) = (f,g), \quad f, g \in \mathcal{H}.$$

In particular, for every $f \in \mathcal{H}$, $\langle \cdot, f\rangle$ is normally distributed with variance $|f|^2$. Thus, from its characteristic function we have

$$\int_{\mathcal{N}'} \exp\left(i\langle x, f\rangle\right) d\mu(x) = \exp\left(-\frac{1}{2}|f|^2\right), \tag{6}$$

which extends (4) and (5) to $f \in \mathcal{H}$.

More generally, for every $n \in \mathbb{N}_0$ and every $f \in \mathcal{H}$, $f \neq 0$, we can derive from the characteristic function (6),

$$\int_{\mathcal{N}'} \langle x, f\rangle^{2n} \, d\mu(x) = \frac{1}{\sqrt{2\pi|f|^2}} \int_{-\infty}^{+\infty} u^{2n} \exp\left(-\frac{u^2}{2|f|^2}\right) du = \frac{(2n)!}{n!2^n}|f|^{2n}$$

$$\int_{\mathcal{N}'} \langle x, f\rangle^{2n+1} \, d\mu(x) = 0$$

and, by the polarization identity,

$$\int_{\mathcal{N}'} \langle x, f_1\rangle \ldots \langle x, f_n\rangle \, d\mu(x)$$

$$= \frac{1}{n!}\sum_{k=1}^{n}(-1)^{n-k} \sum_{i_1<\ldots<i_k} \int_{\mathcal{N}'} \langle x, f_{i_1} + \ldots + f_{i_k}\rangle^n \, d\mu(x),$$

for every $f_1, \ldots, f_n \in \mathcal{H}$, $n \in \mathbb{N}$.

Example 2. Coming back to the white noise space $(S'(\mathbb{R}), \mathcal{C}_\sigma(S'(\mathbb{R})), \mu)$, the previous proposition allows us to consider the Gaussian centered process X with independent increments,

$$X_{1\!\!1_{[0,t)}}(x) = \langle x, 1\!\!1_{[0,t)}\rangle, \quad t \geq 0,$$

being $1\!\!1_B$ the *indicator function* of a Borel set $B \subseteq \mathbb{R}$. This process has covariance

$$(\!(\langle \cdot, 1\!\!1_{[0,t)}\rangle, \langle \cdot, 1\!\!1_{[0,s)}\rangle)\!) = (1\!\!1_{[0,t)}, 1\!\!1_{[0,s)}) = s \wedge t,$$

and thus X is a one-dimensional Brownian motion starting at the origin at time zero. We shall denote this Brownian motion by B and $X_{1\!\!1_{[0,t)}}$ by B_t or $B(t, \cdot)$. Informally, note that

$$B_t(x) = \langle x, 1\!\!1_{[0,t)}\rangle = \int_0^t x(s)\, ds,$$

which suggests considering $x(t)$ as the time derivative of the Brownian motion. Of course, this time derivative does not exist in a pointwise sense. However, it exists as a distribution. From now on, we shall denote $x(t)$ by ω_t or $\omega(t)$ and call it **white noise**. As an aside, let us mention that this example is the connecting point for another direction inside infinite dimensional analysis, the well-known Malliavin Calculus.[36] For a clear explanation about the relation between both infinite dimensional analyses see e.g.[16,38].

Within the more general setting of the Gaussian space

$$(S'_d(\mathbb{R}), \mathcal{C}_\sigma(S'_d(\mathbb{R})), \mu),\ d > 1,$$

we can then introduce a d-dimensional Brownian motion $\mathbf{B}$ starting at the origin at time zero by

$$\mathbf{B}_t(\omega_1, \ldots, \omega_d) := \left(\langle \omega_1, 1\!\!1_{[0,t)}\rangle, \ldots, \langle \omega_d, 1\!\!1_{[0,t)}\rangle\right),\ (\omega_1, \ldots, \omega_d) \in S'_d(\mathbb{R}), t \geq 0.$$

3. Itô-Segal-Wiener Isomorphism

We verify from equalities above Example 2 that the important monomials of the type

$$\begin{aligned}
&\langle \cdot, f\rangle^n = \langle \cdot^{\otimes n}, f^{\otimes n}\rangle,\\
&\langle \cdot, f_1\rangle \ldots \langle \cdot, f_n\rangle = \langle \cdot^{\otimes n}, f_1 \otimes \ldots \otimes f_n\rangle = \langle \cdot^{\otimes n}, f_1 \widehat{\otimes} \ldots \widehat{\otimes} f_n\rangle,
\end{aligned}$$

do not verify an orthogonal relation. This fact is a reason for introducing the orthogonalized so-called Wick-ordered polynomials, a class of functions closely related to the orthogonal Hermite polynomials.

For each $x \in \mathcal{N}'$, let $: x^{\otimes n} :\in \mathcal{N}'^{\widehat{\otimes} n}$, $n \in \mathbb{N}_0$ (Appendix A.1.3) be the so-called *Wick power of order* n, inductively defined by

$$\begin{aligned}
&: x^0 := 1,\\
&: x^1 := x,\\
&: x^{\otimes n} :=: x^{\otimes (n-1)} : \widehat{\otimes} x - (n-1) : x^{\otimes(n-2)} : \widehat{\otimes} \mathrm{Tr}, \quad n \geq 2,
\end{aligned}$$

where $\mathrm{Tr} \in \mathcal{N}'^{\widehat{\otimes} 2}$ is given by

$$\langle \mathrm{Tr}, \xi_1 \otimes \xi_2\rangle = \langle \xi_1, \xi_2\rangle, \quad \xi_1, \xi_2 \in \mathcal{N}.$$

Thus, by induction, for all $x \in \mathcal{N}'$ and all $\xi \in \mathcal{N}$ we have

$$\langle : x^{\otimes n} :, \xi^{\otimes n}\rangle = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} \frac{(2k)!}{k! 2^k} (-\langle \xi, \xi\rangle)^k \langle x, \xi\rangle^{n-2k}, \tag{7}$$

where the right-hand side is the so-called Hermite polynomial in $\langle x,\xi\rangle$ of order n and parameter $\sqrt{\langle\xi,\xi\rangle}=|\xi|$. We recall that given a constant $\sigma>0$, the *n-th Hermite polynomial in $u\in\mathbb{R}$ with parameter σ* is defined by

$$\begin{aligned}: u^n :_{\sigma^2} &:= (-\sigma)^n \exp\left(\frac{u^2}{2\sigma^2}\right)\frac{d^n}{du^n}\exp\left(-\frac{u^2}{2\sigma^2}\right)\\ &= \left(\frac{\sigma}{\sqrt{2}}\right)^n H_n\left(\frac{u}{\sqrt{2}\sigma}\right),\end{aligned}$$

being H_n the *Hermite polynomial of order n*,

$$H_n(u) := (-1)^n \exp\left(u^2\right)\frac{d^n}{du^n}\exp\left(-u^2\right) = 2^n : u^n :_{\frac{1}{2}}, \quad u\in\mathbb{R}, n\in\mathbb{N}_0.$$

That is,

$$H_n(u) = 2^n \sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}\frac{(2k)!}{k!2^k}\left(-\frac{1}{2}\right)^k u^{n-2k}, \quad u\in\mathbb{R}, n\in\mathbb{N}_0.$$

Hence, for each $n\in\mathbb{N}_0$ and every $\xi\in\mathcal{N}$, $\xi\neq 0$, we have

$$\langle : x^{\otimes n} :, \xi^{\otimes n}\rangle =: \langle x,\xi\rangle^n :_{\langle\xi,\xi\rangle} = \left(\frac{|\xi|}{\sqrt{2}}\right)^n H_n\left(\frac{\langle x,\xi\rangle}{\sqrt{2}|\xi|}\right),$$

in accordance with (7). Of course, by the polarization identity, (7) also holds for $\xi\in\mathcal{N}_{\mathbb{C}} := \{\xi_1+i\xi_2 : \xi_1,\xi_2\in\mathcal{N}\}$ with

$$\langle x,\xi_1+i\xi_2\rangle := \langle x,\xi_1\rangle + i\langle x,\xi_2\rangle, \quad x\in\mathcal{N}', \xi_1,\xi_2\in\mathcal{N},$$

meaning that for $f\in\mathcal{H}$ or, more generally, for $f\in\mathcal{H}_{\mathbb{C}}$,

$$\langle f,\xi_1+i\xi_2\rangle = (f,\xi_1) + i(f,\xi_2), \quad \xi_1,\xi_2\in\mathcal{N}.$$

Proposition 2. *For all $\varphi^{(n)}\in\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes}n}$ and all $\phi^{(m)}\in\mathcal{N}_{\mathbb{C}}^{\widehat{\otimes}m}$ the following orthogonal relation holds:*

$$\left(\!\left(\langle : x^{\otimes n} :, \varphi^{(n)}\rangle, \langle : x^{\otimes m} :, \phi^{(m)}\rangle\right)\!\right) = \delta_{n,m} n!(\varphi^{(n)},\phi^{(n)}). \tag{8}$$

Proof. (Sketch) Since elements in $\mathcal{N}^{\widehat{\otimes}n}$, $n\in\mathbb{N}_0$, are linear combinations of elements of the form $\xi^{\otimes n}$ with $\xi\in\mathcal{N}$, it is sufficient to prove (8) for $\varphi^{(n)}$, $\phi^{(m)}$ of the form $\varphi^{(n)}=\xi_1^{\otimes n}$, $\phi^{(m)}=\xi_2^{\otimes m}$, $\xi_1,\xi_2\in\mathcal{N}$. In this case, the proof follows from the orthogonality relation between Hermite polynomials,

$$\int_{-\infty}^{+\infty} H_n(u)H_m(u)\exp\left(-u^2\right) du = \delta_{n,m}\sqrt{\pi}2^n n!,$$

cf. e.g.,[2,14,39]. As before, the general case can then be derived from the real case by means of polarization identity. □

Conversely, since each monomial $u \mapsto u^n$, $n \in \mathbb{N}_0$ can be written as linear combination of Hermite polynomials in $u \in \mathbb{R}$ with any given parameter $\sigma > 0$,

$$u^n = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} \frac{(2k)!}{k!2^k} \sigma^{2k} : u^{n-2k} :_{\sigma^2}, \quad u \in \mathbb{R},$$

then, by the polarization identity, each monomial $\langle \cdot^{\otimes n}, \xi^{\otimes n} \rangle$, $\xi \in \mathcal{N}_{\mathbb{C}}$, can be written as

$$\begin{aligned}\langle x^{\otimes n}, \xi^{\otimes n} \rangle = \langle x, \xi \rangle^n &= \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} \frac{(2k)!}{k!2^k} \langle \xi, \xi \rangle^k : \langle x, \xi \rangle^{n-2k} :_{\langle \xi, \xi \rangle} \\ &= \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} \frac{(2k)!}{k!2^k} \langle \xi, \xi \rangle^k \langle : x^{\otimes(n-2k)} :, \xi^{\otimes(n-2k)} \rangle, \ x \in \mathcal{N}'.\end{aligned}$$

Therefore, the linear space of the so-called *smooth Wick-ordered polynomials*,

$$\mathcal{P}(\mathcal{N}') := \left\{ \Phi : \Phi(x) = \sum_{n=0}^{N} \langle : x^{\otimes n} :, \varphi^{(n)} \rangle, \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}, x \in \mathcal{N}', N \in \mathbb{N}_0 \right\}$$

coincides with the linear space

$$\left\{ \Phi : \Phi(x) = \sum_{n=0}^{N} \langle x^{\otimes n}, \varphi^{(n)} \rangle, \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^{\widehat{\otimes} n}, x \in \mathcal{N}', N \in \mathbb{N}_0 \right\}.$$

In terms of (L^2) properties, it turns out that $\mathcal{P}(\mathcal{N}')$ is dense in (L^2).[48] As a consequence, for any $F \in (L^2)$ there is a sequence $\left(f^{(n)}\right)_{n \in \mathbb{N}_0}$ in the Fock space $\mathrm{Exp}(\mathcal{H}_{\mathbb{C}})$ (Appendix A.1.2) such that

$$F = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} :, f^{(n)} \rangle \tag{9}$$

and, moreover, by the orthogonality property (Proposition 2),

$$\|F\|^2 = \sum_{n=0}^{\infty} n! \left| f^{(n)} \right|^2 = \left\| \left(f^{(n)} \right)_{n \in \mathbb{N}_0} \right\|^2_{\mathrm{Exp}(\mathcal{H}_{\mathbb{C}})}.$$

And vice versa, any series of the form (9) with $\left(f^{(n)}\right)_{n \in \mathbb{N}_0} \in \mathrm{Exp}(\mathcal{H}_{\mathbb{C}})$ defines a function in (L^2). In other words, the expansion (9) yields a unitary isomorphism between the space (L^2) and the symmetric Fock space $\mathrm{Exp}(\mathcal{H}_{\mathbb{C}})$.

Definition 4. We call this unitary isomorphism the Itô-Segal-Wiener isomorphism. The expansion (9) with $\left(f^{(n)}\right)_{n \in \mathbb{N}_0} \in \mathrm{Exp}(\mathcal{H}_{\mathbb{C}})$ is called the Itô-Segal-Wiener chaos decomposition or simply the chaos decomposition of $F \in (L^2)$ and $f^{(n)}$, $n \in \mathbb{N}_0$, the kernels of F.