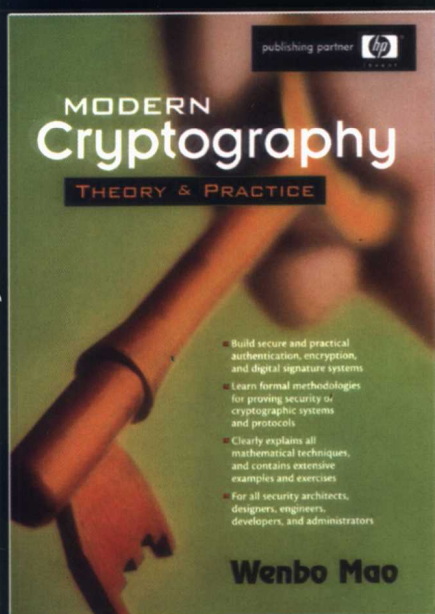


现代密码学 理论与实践

Modern Cryptography:
Theory and Practice



英文版

[英] Wenbo Mao 著



电子工业出版社

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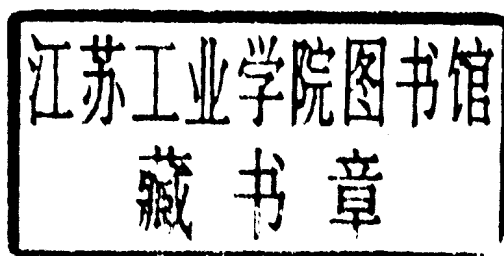
国外计算机科学教材系列

现代密码学理论与实践

(英文版)

Modern Cryptography: Theory and Practice

[英] Wenbo Mao 著



电子工业出版社

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内 容 简 介

第I部分是密码学与信息安全入门性介绍。第II部分介绍学习本书必备的数学背景知识,也可作为学习现代密码学理论基础的系统背景知识。第III部分介绍提供保密和数据完整性保护最基本的密码算法和技术。第IV部分介绍应用密码学和信息安全中一个重要的概念——认证。第V部分对公钥密码技术(加密、签名和签名)的强(实用)安全性概念进行严格的形式化处理,并给出认证协议的形式化分析方法。第VI部分包括两个技术章节和一个简短的评述。

本书适合大学本科生、在高科技公司从事信息安全系统设计和开发的安全工程师、企业信息安全系统管理人员或者生产安全产品的软/硬件开发商以及刚开始从事密码学或计算机安全方面研究的博士生等使用。

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出版说明

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A SHORT DESCRIPTION OF THE BOOK

Many cryptographic schemes and protocols, especially those based on public-key cryptography, have basic or so-called “textbook crypto” versions, as these versions are usually the subjects for many textbooks on cryptography. This book takes a different approach to introducing cryptography: it pays much more attention to *fit-for-application* aspects of cryptography. It explains why “textbook crypto” is only good in an ideal world where data are random and bad guys behave nicely. It reveals the general unfitness of “textbook crypto” for the real world by demonstrating numerous attacks on such schemes, protocols and systems under various real-world application scenarios. This book chooses to introduce a set of practical cryptographic schemes, protocols and systems, many of them standards or de facto ones, studies them closely, explains their working principles, discusses their practical usages, and examines their strong (i.e., fit-for-application) security properties, often with security evidence formally established. The book also includes self-contained theoretical background material that is the foundation for modern cryptography.

PREFACE

Our society has entered an era where commerce activities, business transactions and government services have been, and more and more of them will be, conducted and offered over open computer and communications networks such as the Internet, in particular, via WorldWideWeb-based tools. Doing things online has a great advantage of an always-on availability to people in any corner of the world. Here are a few examples of things that have been, can or will be done online:

Banking, bill payment, home shopping, stock trading, auctions, taxation, gambling, micro-payment (e.g., pay-per-downloading), electronic identity, online access to medical records, virtual private networking, secure data archival and retrieval, certified delivery of documents, fair exchange of sensitive documents, fair signing of contracts, time-stamping, notarization, voting, advertising, licensing, ticket booking, interactive games, digital libraries, digital rights management, pirate tracing, ...

And more can be imagined.

Fascinating commerce activities, transactions and services like these are only possible if communications over open networks can be conducted in a secure manner. An effective solution to securing communications over open networks is to apply cryptography. Encryption, digital signatures, password-based user authentication, are some of the most basic cryptographic techniques for securing communications. However, as we shall witness many times in this book, there are surprising subtleties and serious security consequences in the applications of even the most basic cryptographic techniques. Moreover, for many “fancier” applications, such as many listed in the preceding paragraph, the basic cryptographic techniques are no longer adequate.

With an increasingly large demand for safeguarding communications over open networks for more and more sophisticated forms of electronic commerce, business and services^a, an increasingly large number of information security professionals

^aGartner Group forecasts that total electronic business revenues for business to business (B2B) and business to consumer (B2C) in the European Union will reach a projected US \$2.6 trillion in

will be needed for designing, developing, analyzing and maintaining information security systems and cryptographic protocols. These professionals may range from IT systems administrators, information security engineers and software/hardware systems developers whose products have security requirements, to cryptographers.

In the past few years, the author, a technical consultant on information security and cryptographic systems at Hewlett-Packard Laboratories in Bristol, has witnessed the phenomenon of a progressively increased demand for information security professionals unmatched by an evident shortage of them. As a result, many engineers, who are oriented to application problems and may have little proper training in cryptography and information security have become “roll-up-sleeves” designers and developers for information security systems or cryptographic protocols. This is in spite of the fact that designing cryptographic systems and protocols is a difficult job even for an expert cryptographer.

The author’s job has granted him privileged opportunities to review many information security systems and cryptographic protocols, some of them proposed and designed by “roll-up-sleeves” engineers and are for uses in serious applications. In several occasions, the author observed so-called “textbook crypto” features in such systems, which are the result of applications of cryptographic algorithms and schemes in ways they are usually introduced in many cryptographic textbooks. Direct encryption of a password (a secret number of a small magnitude) under a basic public-key encryption algorithm (e.g., “RSA”) is a typical example of textbook crypto. The appearances of textbook crypto in serious applications with a “non-negligible probability” have caused a concern for the author to realize that the general danger of textbook crypto is not widely known to many people who design and develop information security systems for serious real-world applications.

Motivated by an increasing demand for information security professionals and a belief that their knowledge in cryptography should not be limited to textbook crypto, the author has written this book as a *textbook on non-textbook cryptography*. This book endeavors to:

- Introduce a wide range of cryptographic algorithms, schemes and protocols with a particular emphasis on their *non-textbook* versions.
- Reveal general insecurity of textbook crypto by demonstrating a large number of attacks on and summarizing typical attacking techniques for such systems.
- Provide principles and guidelines for the design, analysis and implementation of cryptographic systems and protocols with a focus on standards.
- Study formalism techniques and methodologies for a rigorous establishment of

2004 (with probability 0.7) which is a 28-fold increase from the level of 2000 [5]. Also, eMarketer (page 41 of [105]) reports that the cost to financial institutions (in USA) due to electronic identity theft was US \$1.4 billion in 2002, and forecasts to grow by a compound annual growth rate of 29%.

strong and fit-for-application security notions for cryptographic systems and protocols.

- Include self-contained and elaborated material as theoretical foundations of modern cryptography for readers who desire a systematic understanding of the subject.

Scope

Modern cryptography is a vast area of study as a result of fast advances made in the past thirty years. This book focuses on one aspect: introducing fit-for-application cryptographic schemes and protocols with their strong security properties evidently established.

The book is organized into the following six parts:

Part I This part contains two chapters (1–2) and serves an elementary-level introduction for the book and the areas of cryptography and information security. Chapter 1 begins with a demonstration on the effectiveness of cryptography in solving a subtle communication problem. A simple cryptographic protocol (first protocol of the book) for achieving “fair coin tossing over telephone” will be presented and discussed. This chapter then carries on to conduct a cultural and “trade” introduction to the areas of study. Chapter 2 uses a series of simple authentication protocols to manifest an unfortunate fact in the areas: pitfalls are everywhere.

As an elementary-level introduction, this part is intended for newcomers to the areas.

Part II This part contains four chapters (3–6) as a set of mathematical background knowledge, facts and basis to serve as a self-contained mathematical reference guide for the book. Readers who only intend to “knowhow,” i.e., know how to use the fit-for-application crypto schemes and protocols, may skip this part yet still be able to follow most contents of the rest of the book. Readers who also want to “know-why,” i.e., know why these schemes and protocols have strong security properties, may find that this self-contained mathematical part is a sufficient reference material. When we present working principles of cryptographic schemes and protocols, reveal insecurity for some of them and reason about security for the rest, it will always be possible for us to refer to a precise point in this part of the book for supporting mathematical foundations.

This part can also be used to conduct a systematic background study of the theoretical foundations for modern cryptography.

Part III This part contains four chapters (7–10) introducing the most basic cryptographic algorithms and techniques for providing privacy and data integrity

protections. Chapter 7 is for symmetric encryption schemes, Chapter 8, asymmetric techniques. Chapter 9 considers an important security quality possessed by the basic and popular asymmetric cryptographic functions when they are used in an ideal world in which data are random. Finally, Chapter 10 covers data integrity techniques.

Since the schemes and techniques introduced here are the most basic ones, many of them are in fact in the textbook crypto category and are consequently *insecure*. While the schemes are introduced, abundant attacks on many schemes will be demonstrated with warning remarks explicitly stated. For practitioners who do not plan to proceed with an in-depth study of fit-for-application crypto and their strong security notions, this textbook crypto part will still provide these readers with explicit early warning signals on the general insecurity of textbook crypto.

Part IV This part contains three chapters (11–13) introducing an important notion in applied cryptography and information security: authentication. These chapters provide a wide coverage of the topic. Chapter 11 includes technical background, principles, a series of basic protocols and standards, common attacking tricks and prevention measures. Chapter 12 is a case study for four well-known authentication protocol systems for real world applications. Chapter 13 introduces techniques which are particularly suitable for open systems which cover up-to-date and novel techniques.

Practitioners, such as information security systems administration staff in an enterprise and software/hardware developers whose products have security consequences may find this part helpful.

Part V This part contains four chapters (14–17) which provide formalism and rigorous treatments for strong (i.e., fit-for-application) security notions for public-key cryptographic techniques (encryption, signature and signcryption) and formal methodologies for the analysis of authentication protocols. Chapter 14 introduces formal definitions of strong security notions. The next two chapters are fit-for-application counterparts to textbook crypto schemes introduced in Part III, with strong security properties formally established (i.e., evidently reasoned). Finally, Chapter 17 introduces formal analysis methodologies and techniques for the analysis of authentication protocols, which we have not been able to deal with in Part IV.

Part VI This is the final part of the book. It contains two technical chapters (18–19) and a short final remark (Chapter 20). The main technical content of this part, Chapter 18, introduces a class of cryptographic protocols called zero-knowledge protocols. These protocols provide an important security service which is needed in various “fancy” electronic commerce and business applications: verification of a claimed property of secret data (e.g., in conforming with a business requirement) while preserving a strict privacy quality for the

claimant. Zero-knowledge protocols to be introduced in this part exemplify the diversity of special security needs in various real world applications, which are beyond confidentiality, integrity, authentication and non-repudiation. In the final technical chapter of the book (Chapter 19) we will complete our job which has been left over from the first protocol of the book: to realize “fair coin tossing over telephone.” That final realization will achieve a protocol which has evidently-established strong security properties yet with an efficiency suitable for practical applications.

Needless to say, a description for each fit-for-application crypto scheme or protocol has to begin with a reason why the textbook crypto counterpart is unfit for application. Invariably, these reasons are demonstrated by attacks on these schemes or protocols, which, by the nature of attacks, often contain a certain degree of subtleties. In addition, a description of a fit-for-application scheme or protocol must also end at an analysis that the strong (i.e., fit-for-application) security properties do hold as claimed. Consequently, some parts of this book inevitably contain mathematical and logical reasonings, deductions and transformations in order to manifest attacks and fixes.

While admittedly fit-for-application cryptography is not a topic for quick mastery or that can be mastered via light reading, this book, nonetheless, is not one for in-depth research topics which will only be of interest to specialist cryptographers. The things reported and explained in it are well-known and quite elementary to cryptographers. The author believes that they can also be comprehended by non-specialists if the introduction to the subject is provided with plenty of explanations and examples and is supported by self-contained mathematical background and reference material.

The book is aimed at the following readers.

- Students who have completed, or are near to completion of, first degree courses in computer, information science or applied mathematics, and plan to pursue a career in information security. For them, this book may serve as an advanced course in applied cryptography.
- Security engineers in high-tech companies who are responsible for the design and development of information security systems. If we say that the consequence of textbook crypto appearing in an academic research proposal may not be too harmful since the worst case of the consequence would be an embarrassment, then the use of textbook crypto in an information security product may lead to a serious loss. Therefore, knowing the unfitness of textbook crypto for real world applications is necessary for these readers. Moreover, these readers should have a good understanding of the security principles behind the fit-for-application schemes and protocols and so they can apply the schemes and the principles correctly. The self-contained mathematical foundations material in Part II makes the book a suitable self-teaching text for

these readers.

- Information security systems administration staff in an enterprise and software/hardware systems developers whose products have security consequences. For these readers, Part I is a simple and essential course for cultural and “trade” training; Parts III and IV form a suitable cut-down set of knowledge in cryptography and information security. These three parts contain many basic crypto schemes and protocols accompanied with plenty of attacking tricks and prevention measures which should be known to and can be grasped by this population of readers without demanding them to be burdened by theoretical foundations.
- New Ph.D. candidates beginning their research in cryptography or computer security. These readers will appreciate a single-point reference book which covers formal treatment of strong security notions and elaborates these notions adequately. Such a book can help them to quickly enter into the vast area of study. For them, Parts II, IV, V and VI constitute a suitable level of literature survey material which can lead them to find further literatures, and can help them to shape and specialize their own research topics.
- A cut-down subset of the book (e.g., Part I, II, III and VI) also form a suitable course in applied cryptography for undergraduate students in computer science, information science and applied mathematics courses.

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Please send suggestions and corrections to the author (wenbo.mao@hp.com). Many thanks! Corrections will be listed on the website for the book:

www-uk.hpl.hp.com/people/wm/mctp.html

Bristol, England

May 2003

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